Graphs II

Centrality, and algorithms you should know

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Partly based on materials by
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Centrality

= “Importance”

Why Node Centrality?

What can we do if we can rank all the nodes in a graph (e.g., Facebook, LinkedIn, Twitter)?

- Find **celebrities** or influential people in a social network (Twitter)
- Find "**gatekeepers**" who connect communities (headhunters love to find them on LinkedIn)
- What else?

More generally

Helps **graph analysis, visualization, understanding**, e.g.,

- let us rank nodes, group or study them by centrality
- only show subgraph formed by the **top 100 nodes**, out of the millions in the full graph
    - similar to google search results (ranked, and they only show you 10 per page)

Can also compute edge centrality.
Here we focus on node centrality.

Degree Centrality (easiest)

Degree = number of neighbors

For directed graphs

- in degree = # incoming edges
- out degree = # outgoing edges

Algorithms?

- Sequential scan through edge list
- What about for a graph stored in SQLite?

Computing degrees using SQL

Recall simplest way to store a graph in SQLite:

```
edges(source_id, target_id)
```

1. Create index for each column

2. Use **group by** statement to find node degrees

```
select count(*) from edges group by source_id;
```

Betweenness Centrality

High betweenness

- = important "gatekeeper" or liaison

A node's betweenness

- = $\sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}$

Number of shortest paths between s and t that involves v

Number of shortest paths between s and t

Needs to compute all-pairs shortest path (O(N³)). Slow.

Clustering Coefficient

Technically not a centrality measure, but useful

A node's clustering coefficient is a measure of how close the node's neighbors are from forming a clique.

- Value of 1 = neighbors form a clique
- Value of 0 = No edges among neighbors

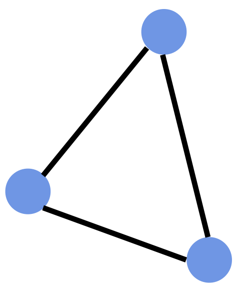
(Assuming undirected graph)

Computing Clustering Coefficient...

Requires triangle counting

Real social networks have a lot of triangles

- Friends of friends are friends



But: triangles are expensive to compute

(3-way join; several approx. algos)

Can we do that quickly?

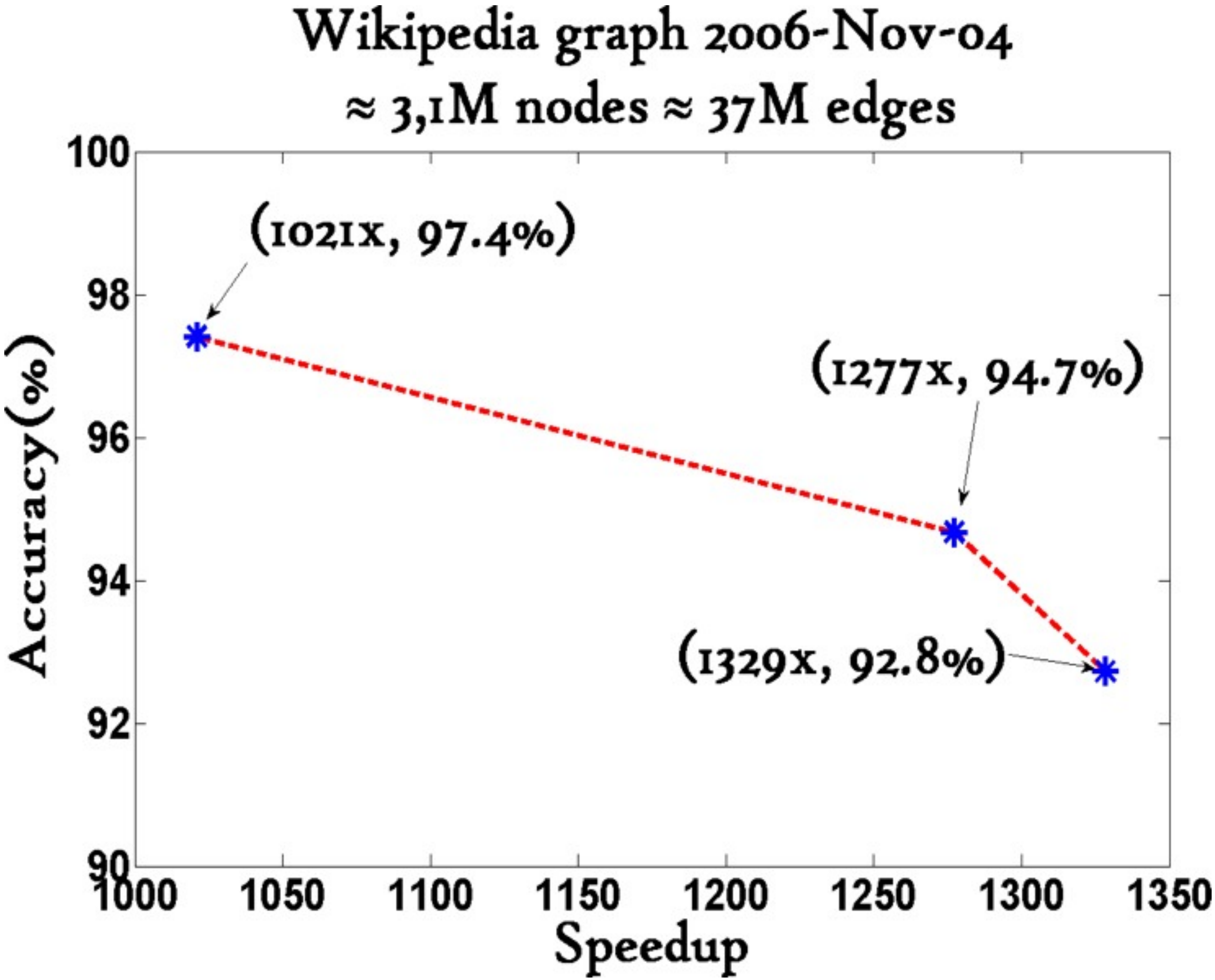
details

Super Fast Triangle Counting
[Tsourakakis ICDM 2008]

But: triangles are expensive to compute
(3-way join; several approx. algos)
Q: Can we do that quickly?
A: Yes!

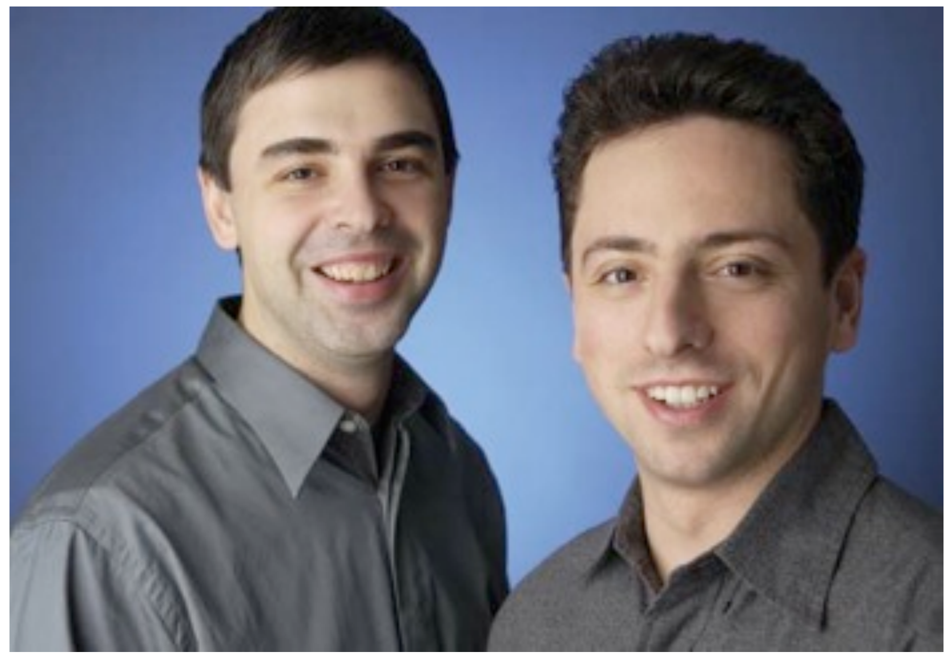
#triangles = 1/6 Sum ($(\lambda i)^3$)

(and, because of skewness,
we only need the top few eigenvalues!



1000x+ speed-up, >90% accuracy

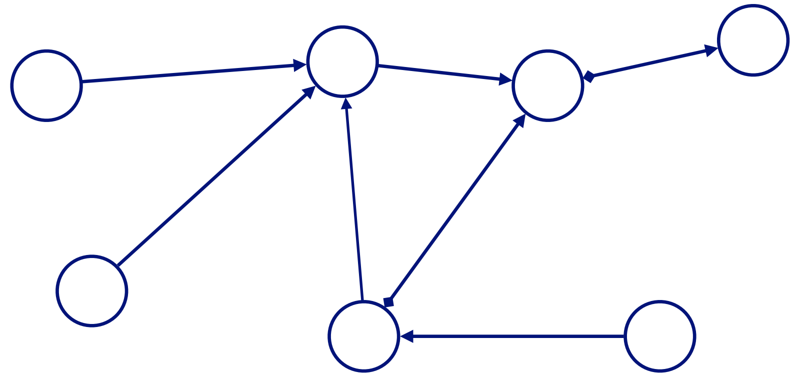
PageRank (google)

Larry Page  Sergey Brin

Brin, Sergey and Lawrence Page (1998). *Anatomy of a Large-Scale Hypertextual Web Search Engine*. 7th Intl World Wide Web Conf.

Problem: PageRank

Given a directed graph, find its most interesting/central node

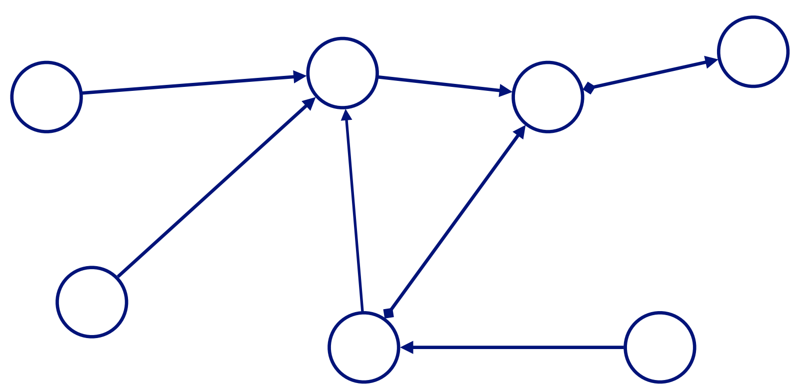


A node is important, if it is connected with important nodes (recursive, but OK!)

Problem: PageRank - solution

Given a directed graph, find its most interesting/central node

Proposed solution: Random walk; spot most ‘popular’ node (-> steady state prob. (ssp))

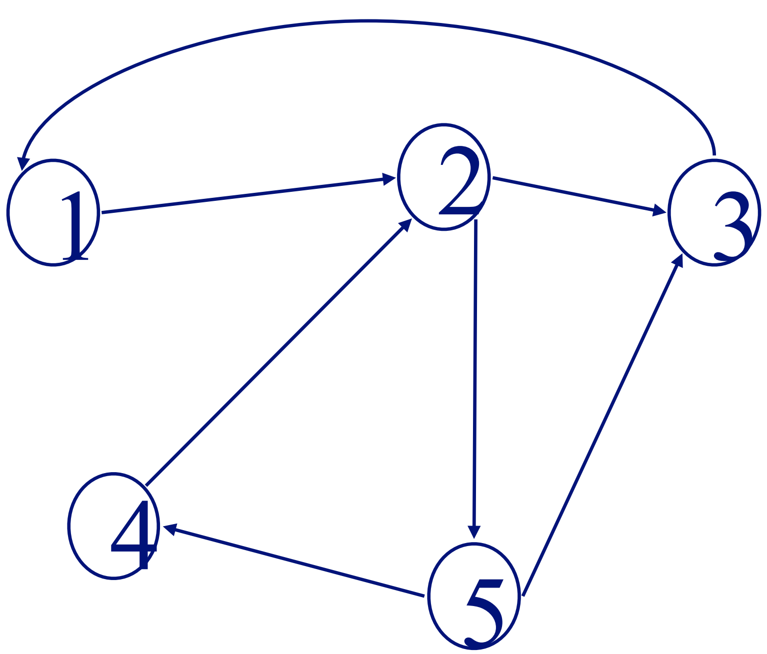


“state” = webpage

A node has high ssp, if it is connected with high ssp nodes (recursive, but OK!)

(Simplified) PageRank

- Let $\mathbf{A}$ be the adjacency matrix;
- let $\mathbf{B}$ be the transition matrix: transpose, column-normalized - then



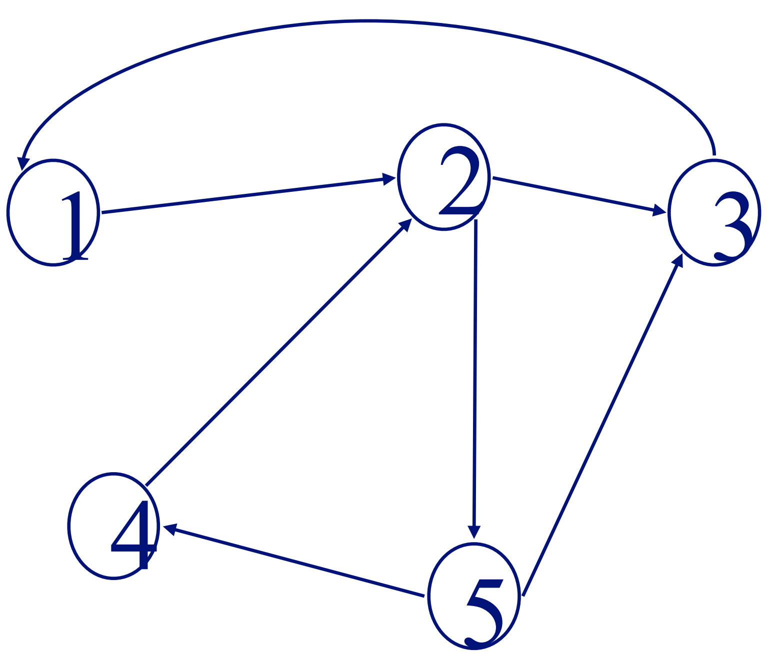
To \ From

$$\mathbf{B}$$

$$\begin{bmatrix} & & 1 & & \\ 1 & & & 1 & \\ & 1/2 & & & 1/2 \\ & & & & 1/2 \\ & 1/2 & & & \end{bmatrix} \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix} = \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix}$$

(Simplified) PageRank

- **B p = p**



$$\overset{\mathbf{B}}{\begin{bmatrix} & & 1 & & \\ 1 & & & 1 & \\ & 1/2 & & & 1/2 \\ & & & & 1/2 \\ & 1/2 & & & \end{bmatrix}} \overset{\mathbf{p}}{\begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix}} \overset{=}{=} \overset{\mathbf{p}}{\begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix}}$$

(Simplified) PageRank

- $\mathbf{B}\,\mathbf{p} = 1 * \mathbf{p}$
- thus, $\mathbf{p}$ is the **eigenvector** that corresponds to the highest eigenvalue (=1, since the matrix is column-normalized)
- Why does such a $\mathbf{p}$ exist?
  - $\mathbf{p}$ exists if $\mathbf{B}$ is nxn, nonnegative, irreducible [Perron–Frobenius theorem]

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(Simplified) PageRank

- In short: imagine a particle randomly moving along the edges
- compute its steady-state probabilities (ssp)

Full version of algo:
  with occasional random jumps

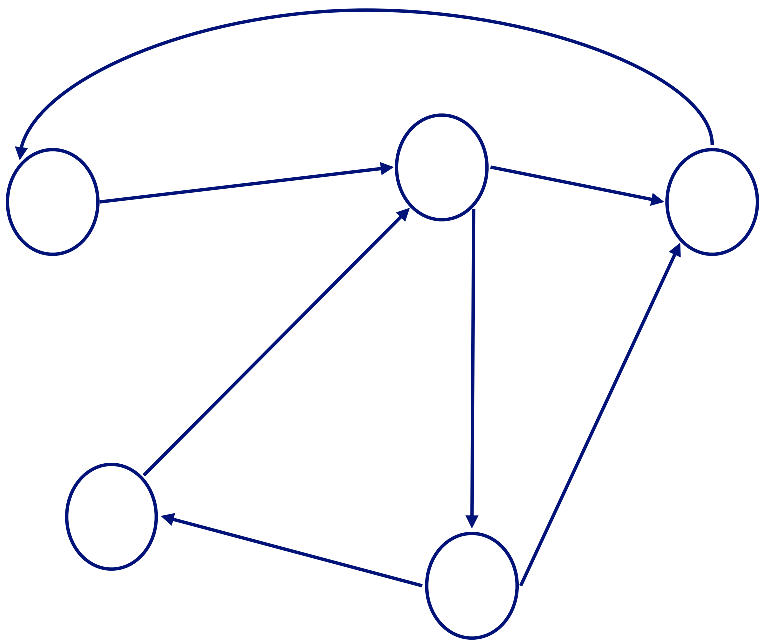
Why? To make the matrix irreducible

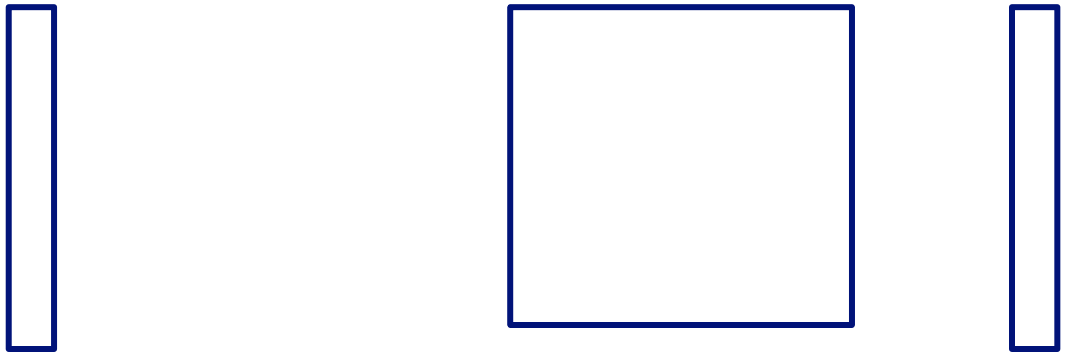
Full Algorithm

- With probability *1-c*, fly-out to a random node
- Then, we have

  $\mathbf{p} = c\,\mathbf{B}\,\mathbf{p} + (1-c)/n\;\mathbf{1} \Rightarrow$

  $\mathbf{p} = (1-c)/n\;[\mathbf{I} - c\,\mathbf{B}]^{-1}\;\mathbf{1}$





Full Algorithm

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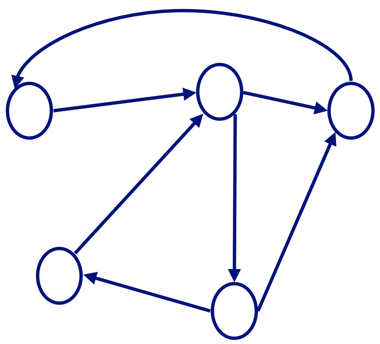

$\mathbf{p} = c\ \mathbf{B}\ \mathbf{p} + (1\text{-}c)/n\ \mathbf{1} \Rightarrow$

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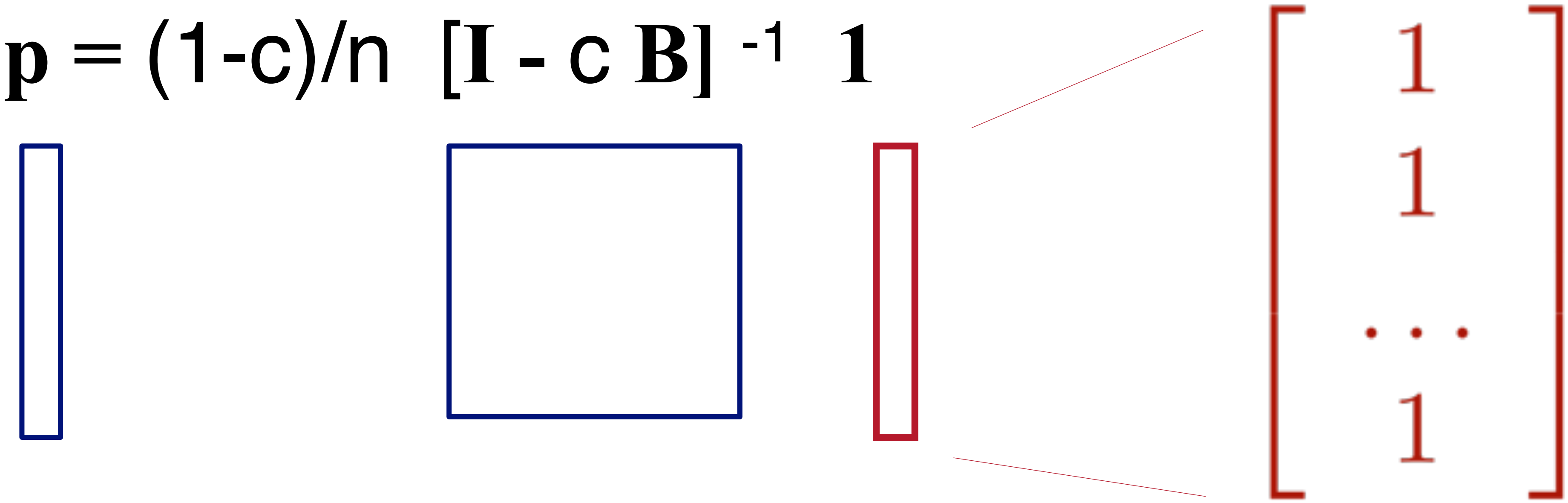


Alternative notation – eigenvector viewpoint

$\mathbf{M}$ Modified transition matrix

$$\mathbf{M} = c\,\mathbf{B} + (1-c)/n \;\; \mathbf{1} \;\; \mathbf{1}^T$$

Then

$$\mathbf{p} = \mathbf{M}\,\mathbf{p}$$

That is: the steady state probabilities =

PageRank scores form the *first eigenvector* of the 'modified transition matrix'

PageRank for graphs (generally)

You can compute PageRank for **any graphs**

Should be in your algorithm “toolbox”

- Better than simple centrality measure (e.g., degree)
- Fast to compute for large graphs (O(E))

But can be “misled” (Google Bomb)

- How?

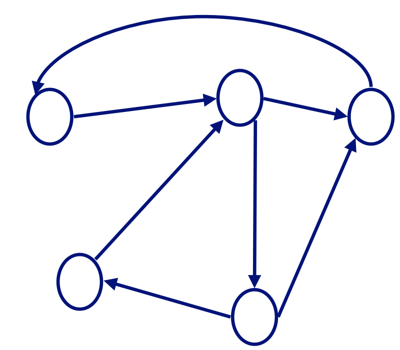
Personalized PageRank

Make one small variation of PageRank

- Intuition: not all pages are equal, some more relevant to a person’s specific needs
- How?

“Personalizing” PageRank

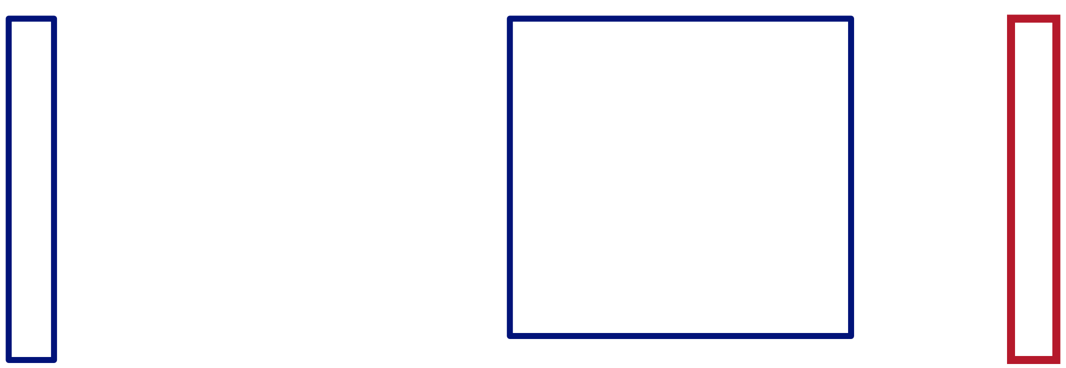
- With probability $1\text{-}c$, fly-out to ~~a random node~~ **some preferred nodes**



- Then, we have

$$\mathbf{p} = c\,\mathbf{B}\,\mathbf{p} + (1\text{-}c)/n\ \mathbf{1} \Rightarrow$$

$$\mathbf{p} = (1\text{-}c)/n\ [\mathbf{I} - c\,\mathbf{B}]^{-1}\ \mathbf{1}$$



$$\begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}$$

Why learn about Personalized PageRank?

Can be used for recommendation, e.g.,

- If I like this webpage, what would I also be interested?
- If I like this product, what other products I also like? (in a user-product bipartite graph)

Again, very flexible. Can be run on any graph

Will see some interactive tools in next lecture that uses Personalized PageRank