

<http://poloclub.gatech.edu/cse6242>

CSE6242 / CX4242: Data & Visual Analytics

# Time Series

Mining and Forecasting

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Partly based on materials by

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# Outline

- ➔ • Motivation
- Similarity search – distance functions
- Linear Forecasting
- Non-linear forecasting
- Conclusions

# Problem definition

- **Given:** one or more sequences

$x_1, x_2, \dots, x_t, \dots$

$(y_1, y_2, \dots, y_t, \dots)$

$(\dots)$

- **Find**
  - similar sequences; forecasts
  - patterns; clusters; outliers

# Motivation - Applications

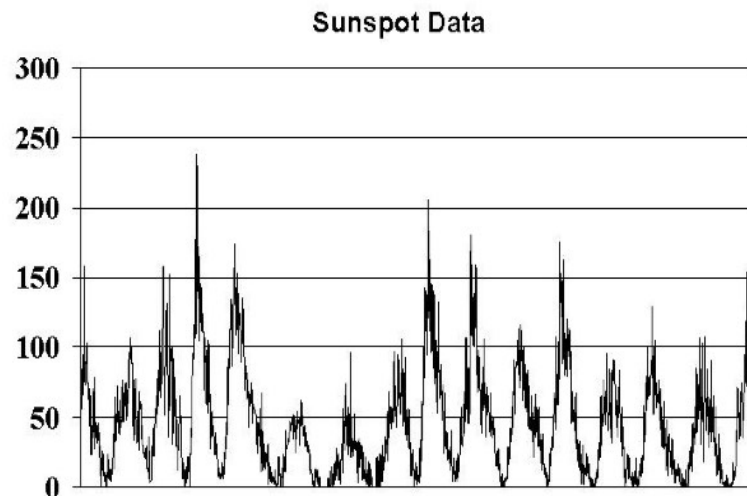
- Financial, sales, economic series
- Medical
  - ECGs +; blood pressure etc monitoring
  - reactions to new drugs
  - elderly care

# Motivation - Applications (cont'd)

- 'Smart house'
  - sensors monitor temperature, humidity, air quality
- video surveillance

# Motivation - Applications (cont'd)

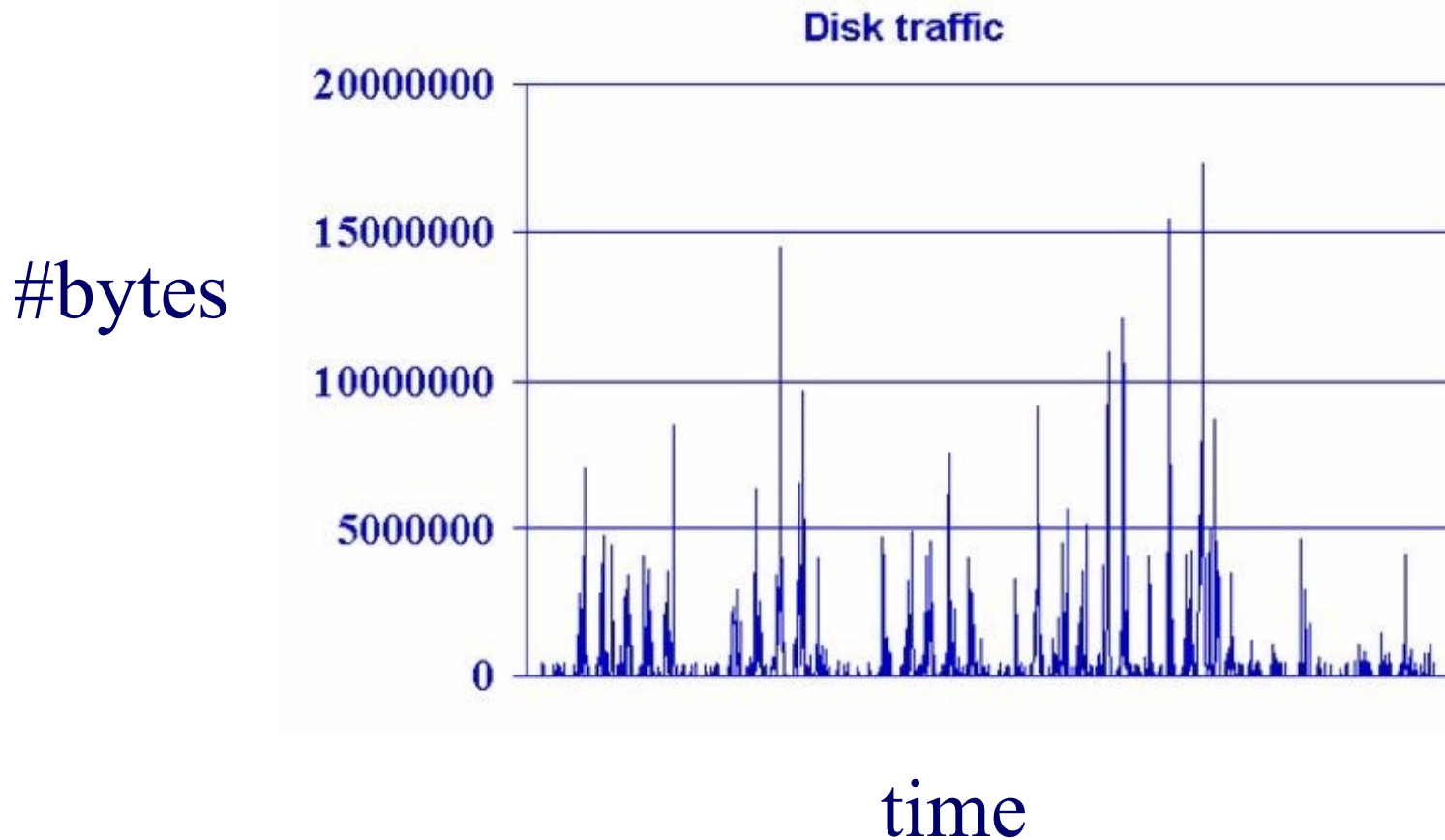
- Weather, environment/anti-pollution
  - volcano monitoring
  - air/water pollutant monitoring



# Motivation - Applications (cont'd)

- Computer systems
  - ‘Active Disks’ (buffering, prefetching)
  - web servers (ditto)
  - network traffic monitoring
  - ...

# Stream Data: Disk accesses



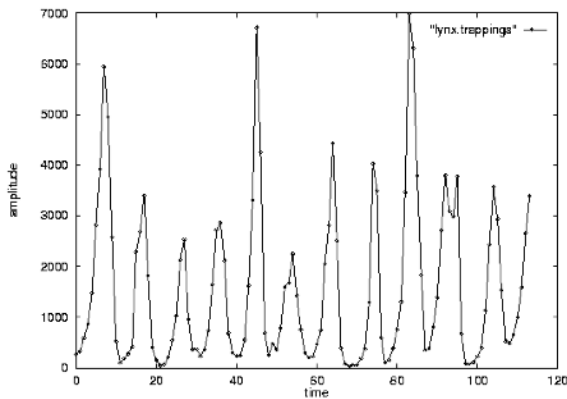


# Problem #1:

**Goal:** given a signal (e.g., #packets over time)

**Find:** patterns, periodicities, and/or compress

count

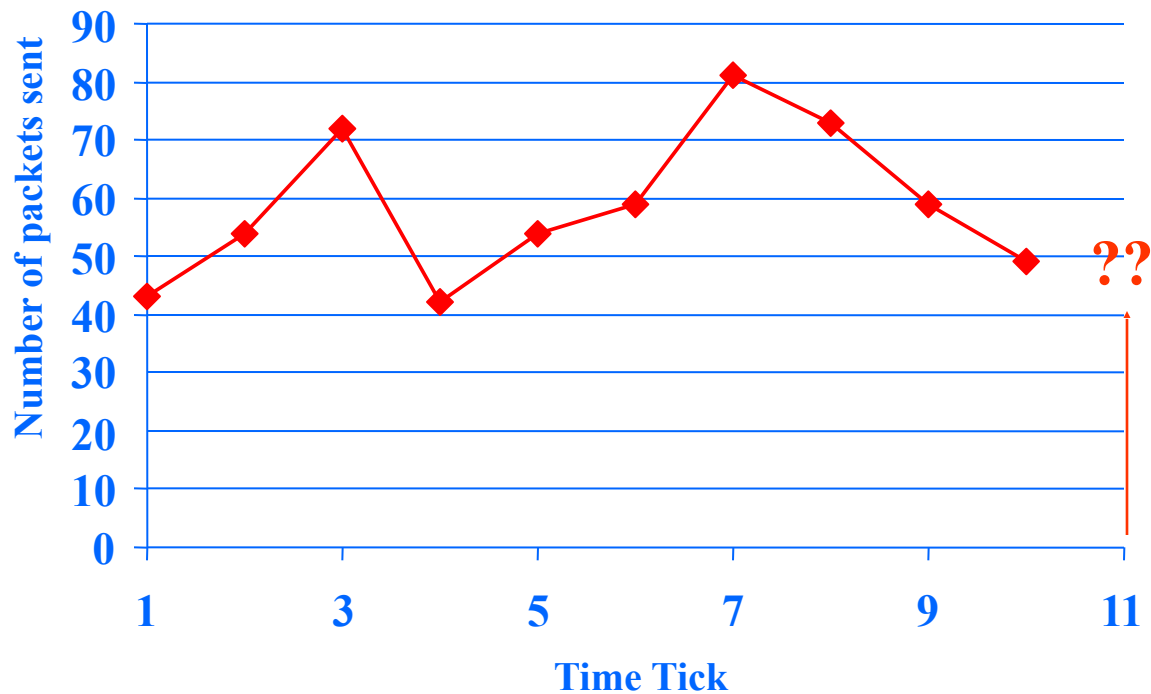


year

lynx caught per year  
(packets per day;  
temperature per day)

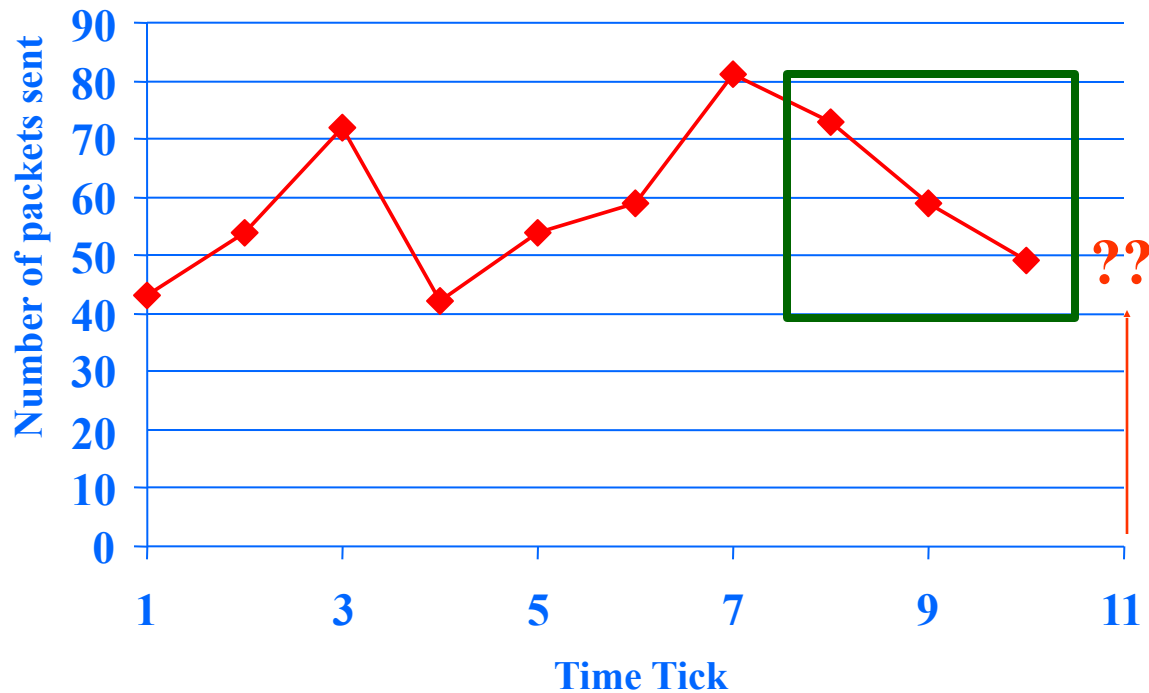
# Problem#2: Forecast

Given  $x_t, x_{t-1}, \dots$ , forecast  $x_{t+1}$



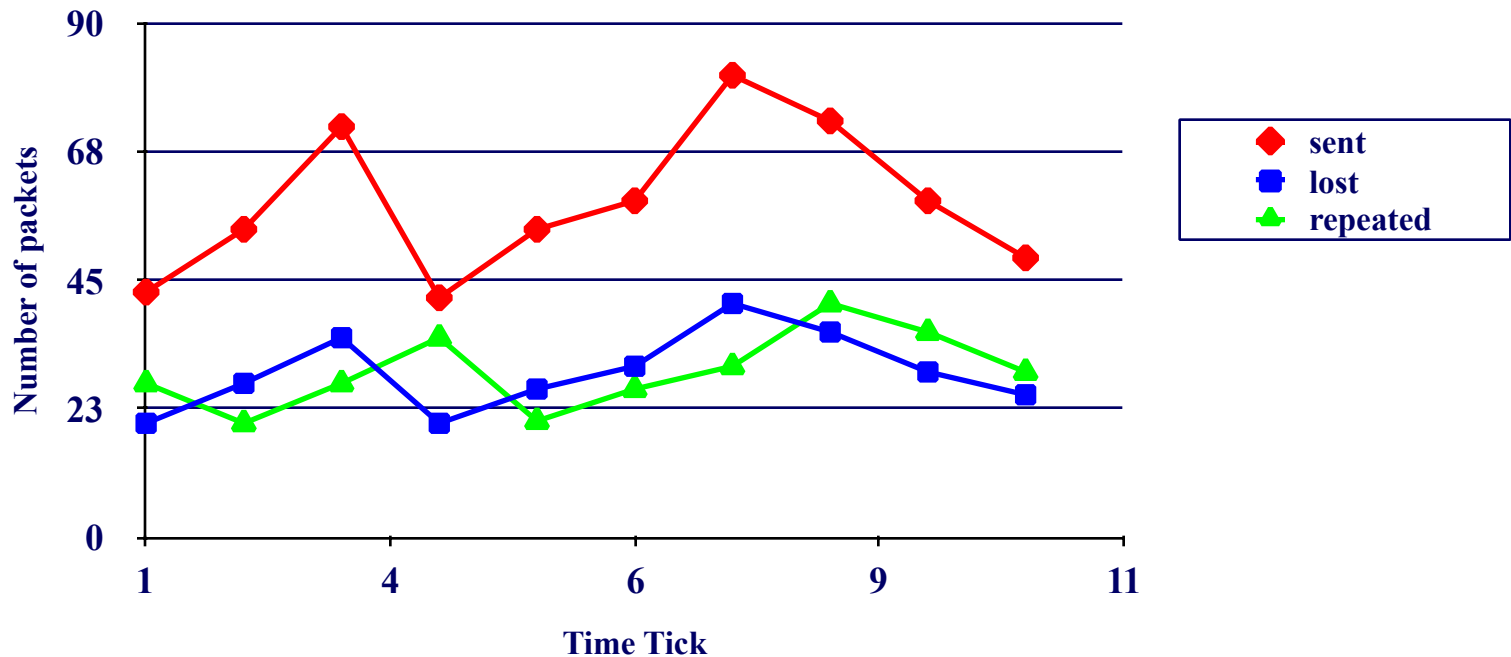
# Problem#2': Similarity search

E.g., Find a 3-tick pattern, similar to the last one



# Problem #3:

- Given: A set of **correlated** time sequences
- Forecast 'Sent(t)'



# Important observations

Patterns, rules, forecasting and similarity indexing are closely related:

- To do forecasting, we need
  - to find patterns/rules
  - to find similar settings in the past
- to find outliers, we need to have forecasts
  - (outlier = too far away from our forecast)

# Outline

- Motivation
- ➔ • Similarity search and distance functions
  - Euclidean
  - Time-warping
- ...

# Importance of distance functions

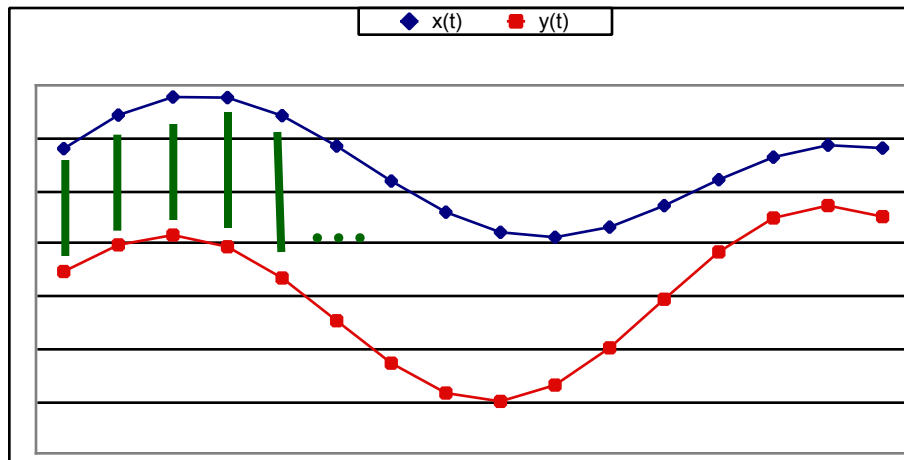
Subtle, but **absolutely necessary**:

- A ‘must’ for similarity indexing (-> forecasting)
- A ‘must’ for clustering

Two major families

- Euclidean and  $L_p$  norms
- Time warping and variations

# Euclidean and Lp



$$D(\vec{x}, \vec{y}) = \sum_{i=1}^n (x_i - y_i)^2$$

$$L_p(\vec{x}, \vec{y}) = \sum_{i=1}^n |x_i - y_i|^p$$

$L_1$ : city-block = Manhattan

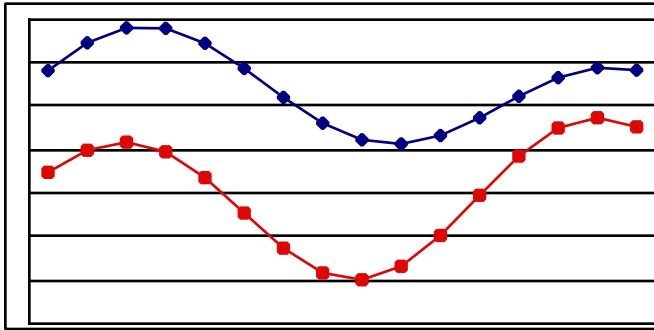
$L_2$  = Euclidean

$L_\infty$

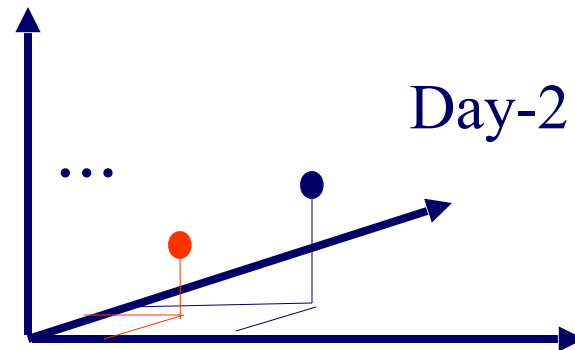


# Observation #1

Time sequence  $\rightarrow$  n-d vector



Day-n

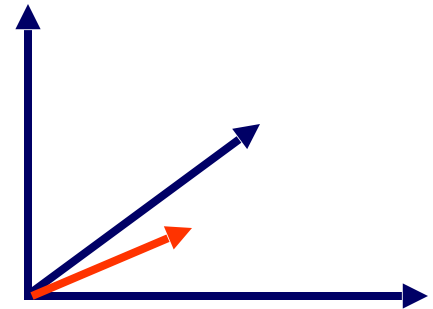
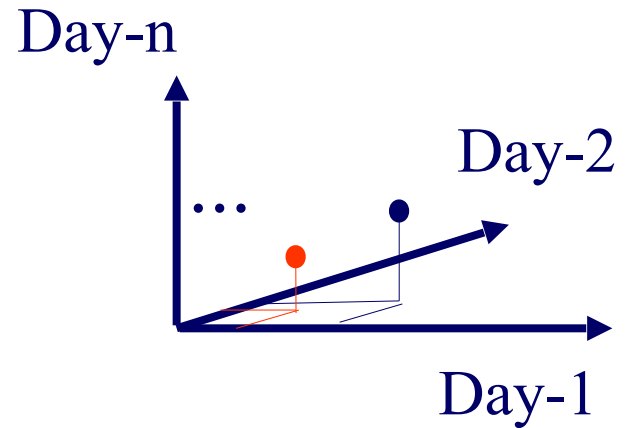


Day-1

# Observation #2

Euclidean distance is closely related to

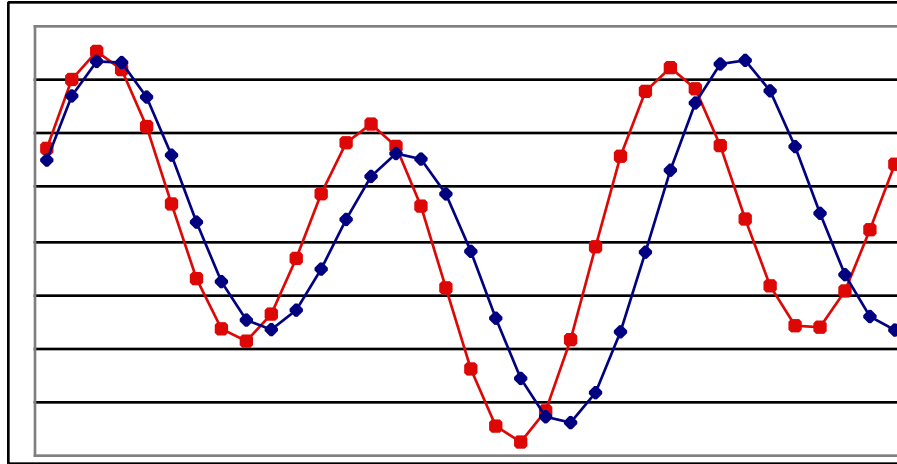
- cosine similarity
- dot product



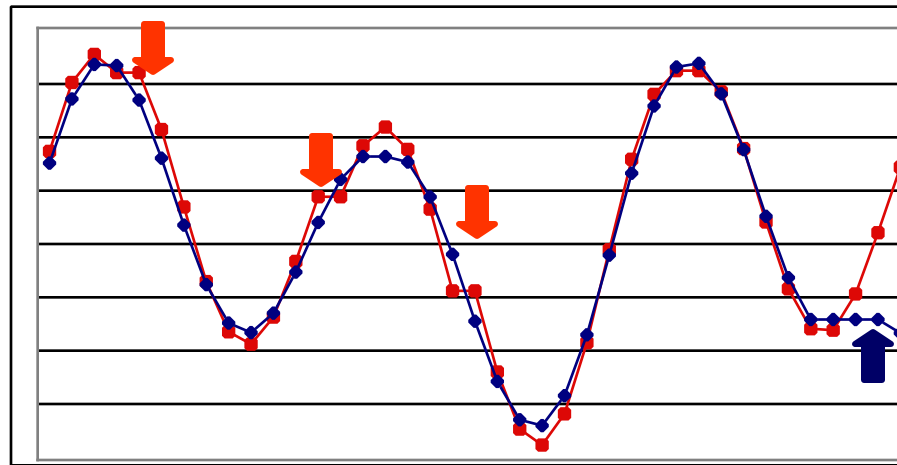
# Time Warping

- allow accelerations - decelerations
  - (with or without penalty)
- THEN compute the (Euclidean) distance (+ penalty)
- related to the string-editing distance

# Time Warping



‘stutters’:



# Time warping

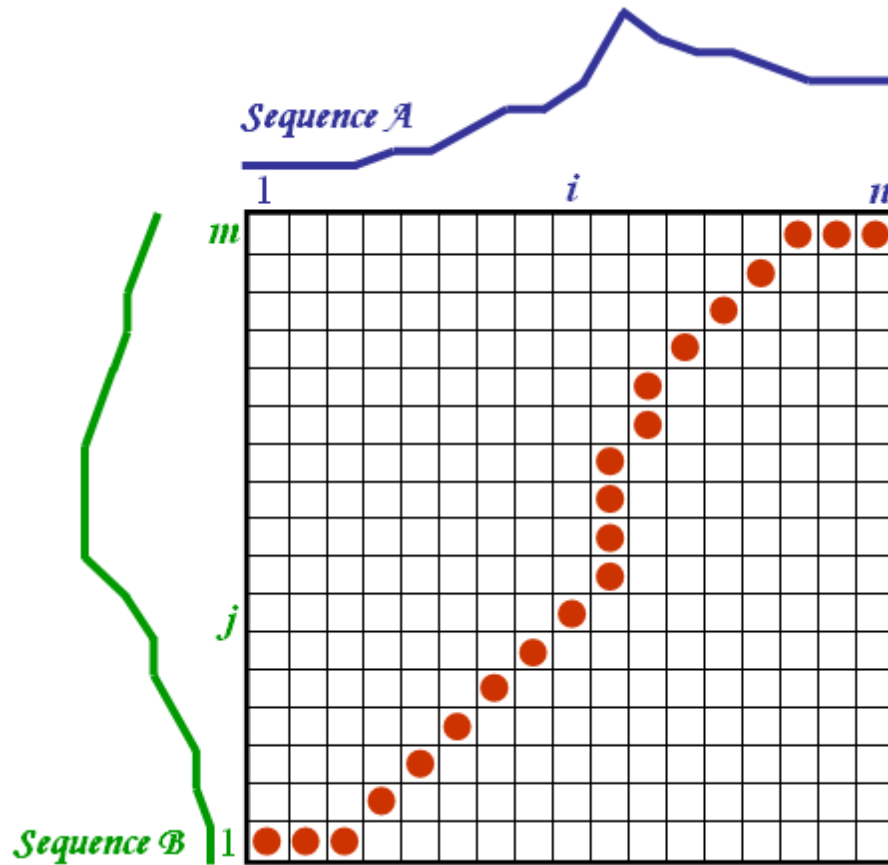
Q: how to compute it?

A: dynamic programming

$D(i, j) = \text{cost to match}$

prefix of length  $i$  of first sequence  $x$  with prefix  
of length  $j$  of second sequence  $y$

# Time warping



# Time warping

Thus, with no penalty for stutter, for sequences

$$x_1, x_2, \dots, x_i, \quad y_1, y_2, \dots, y_j$$

$$D(i, j) = \|x[i] - y[j]\| + \min \begin{cases} D(i-1, j-1) & \text{no stutter} \\ D(i, j-1) & \text{x-stutter} \\ D(i-1, j) & \text{y-stutter} \end{cases}$$

# Time warping

VERY SIMILAR to the string-editing distance

$$D(i, j) = \|x[i] - y[j]\| + \min \begin{cases} D(i-1, j-1) & \text{no stutter} \\ D(i, j-1) & \text{x-stutter} \\ D(i-1, j) & \text{y-stutter} \end{cases}$$



# Time warping

- Complexity:  $O(M*N)$  - quadratic on the length of the strings
- Many variations (penalty for stutters; limit on the number/percentage of stutters; ...)
- popular in voice processing  
[Rabiner + Juang]

# Other Distance functions

- piece-wise linear/flat approx.; compare pieces [Keogh+01] [Faloutsos+97]
- ‘cepstrum’ (for voice [Rabiner+Juang])
  - do DFT; take log of amplitude; do DFT again!
- Allow for small gaps [Agrawal+95]

See tutorial by [Gunopulos + Das,  
SIGMOD01]

# Other Distance functions


- In [Keogh+, KDD'04]: parameter-free, MDL based

# Conclusions

Prevailing distances:

- Euclidean and
- time-warping

# Outline

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- Similarity search and distance functions
-  • Linear Forecasting
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# Linear Forecasting

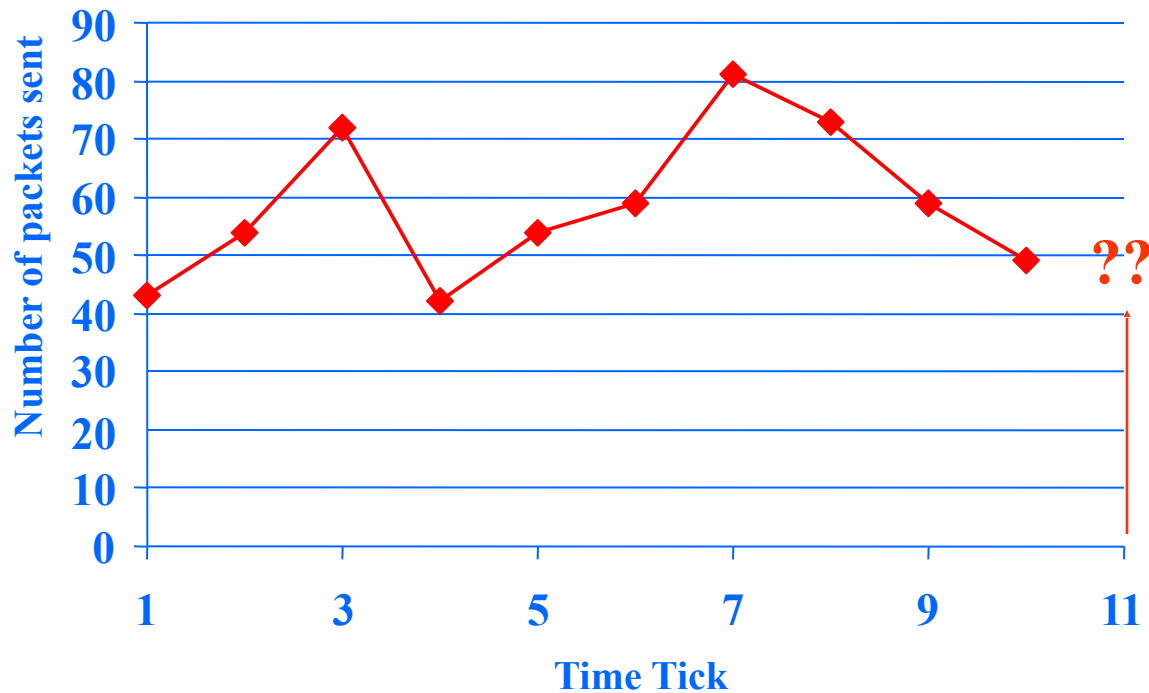
# Outline

- Motivation
- ...
- Linear Forecasting
  - Auto-regression: Least Squares; RLS
  - Co-evolving time sequences
  - Examples
  - Conclusions



# Problem#2: Forecast

- Example: give  $x_{t-1}, x_{t-2}, \dots$ , forecast  $x_t$





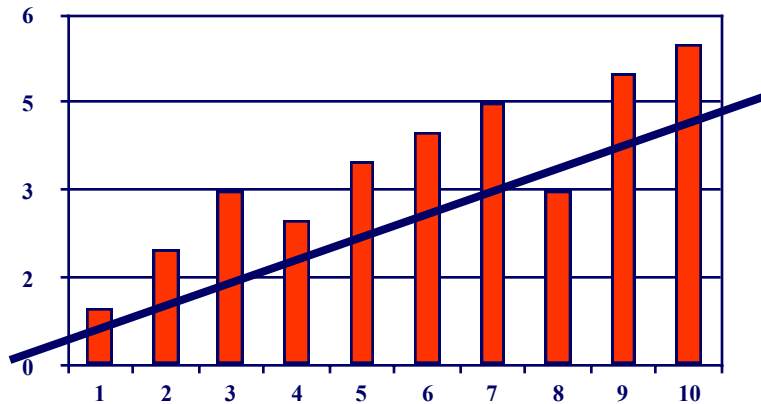
# Forecasting: Preprocessing

MANUALLY:

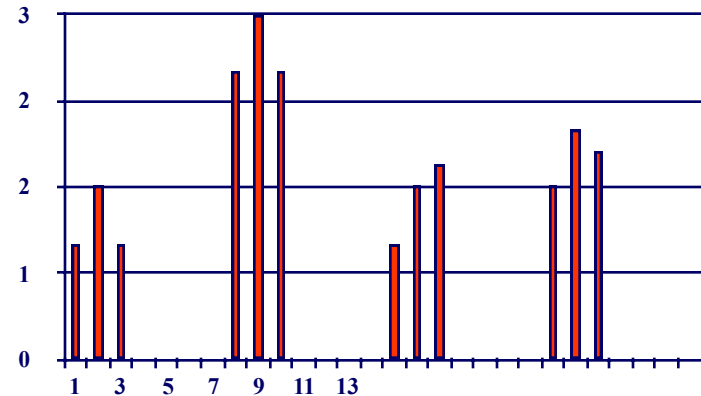
remove trends

spot periodicities

7 days



time



time

# Problem#2: Forecast

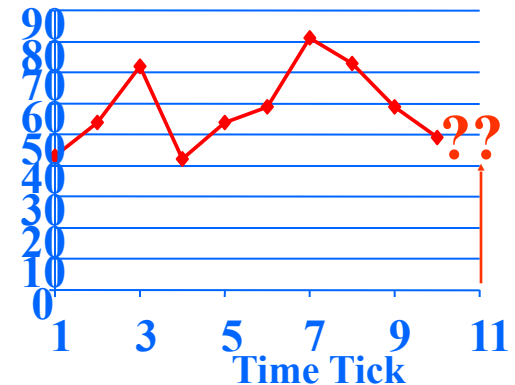
- Solution: try to express

$x_t$

as a linear function of the past:  $x_{t-1}, x_{t-2}, \dots$ ,  
(up to a window of  $w$ )

Formally:

$$x_t \approx a_1 x_{t-1} + \dots + a_w x_{t-w} + noise$$



# (Problem: Back-cast; interpolate)

- Solution - interpolate: try to express

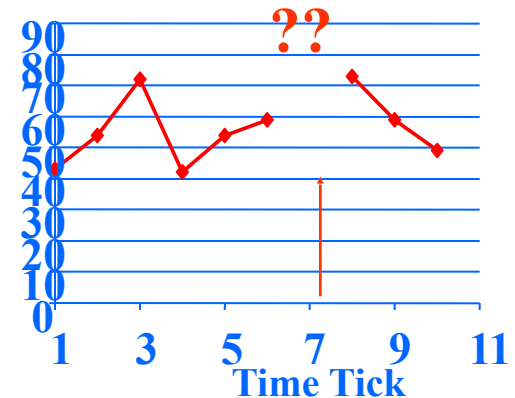
$x_t$

as a linear function of the past AND the future:

$x_{t+1}, x_{t+2}, \dots, x_{t+w_{future}}; x_{t-1}, \dots, x_{t-w_{past}}$

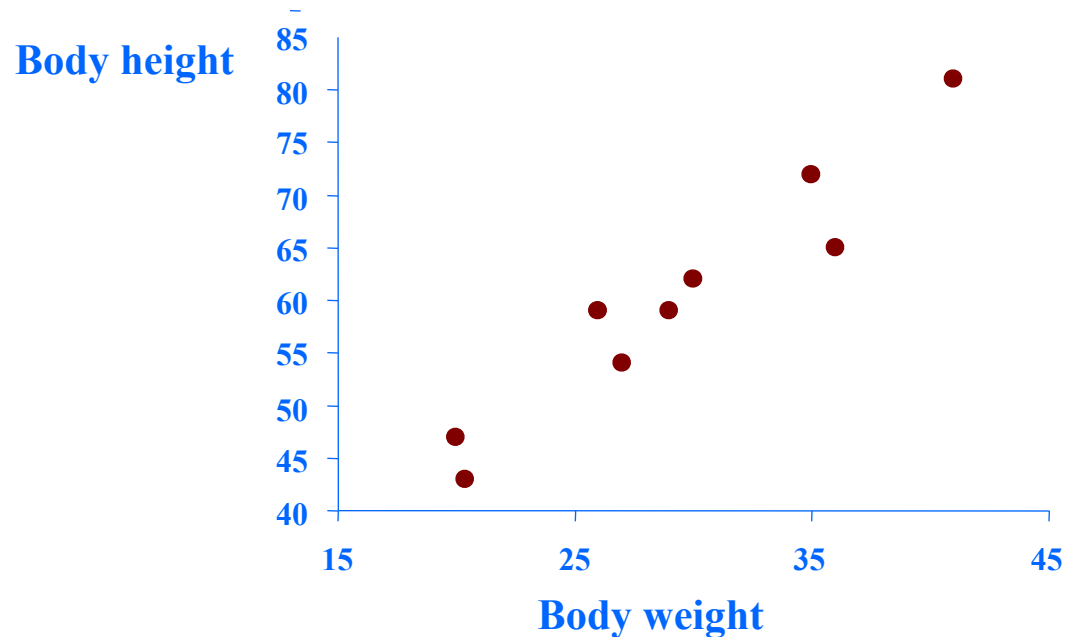
(up to windows of  $w_{past}$ ,  $w_{future}$ )

- EXACTLY the same algo's



# Refresher: Linear Regression

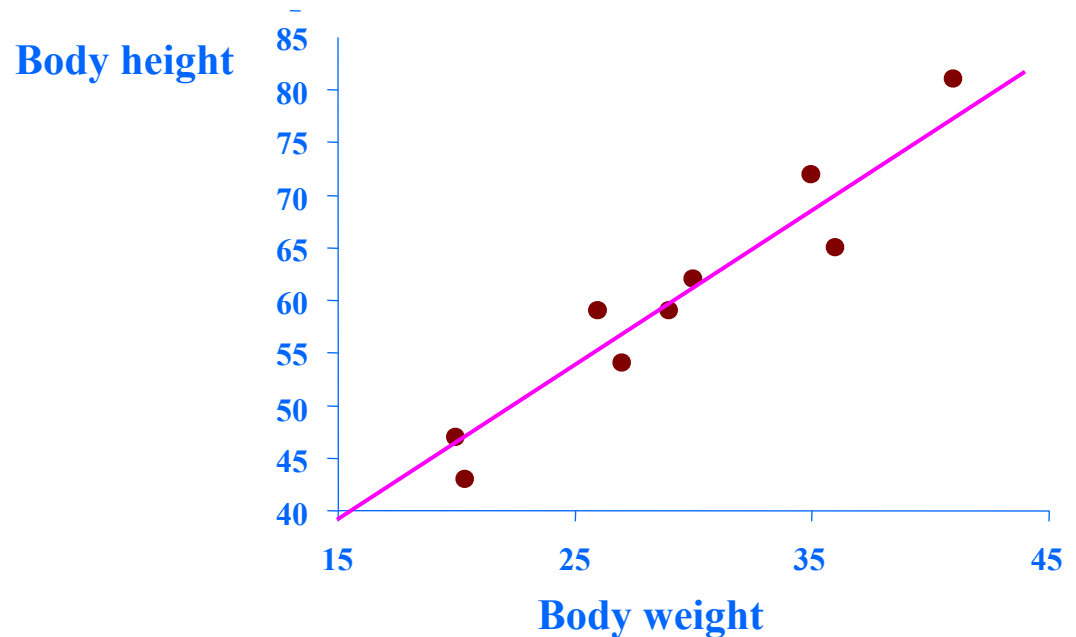
<i>patient</i>	<i>weight</i>	<i>height</i>
1	27	43
2	43	54
3	54	72
...	...	...
N	25	??



Express what we **don't know** (= “dependent variable”)  
as a linear function of what we **know** (= “independent variable(s)”)

# Refresher: Linear Regression

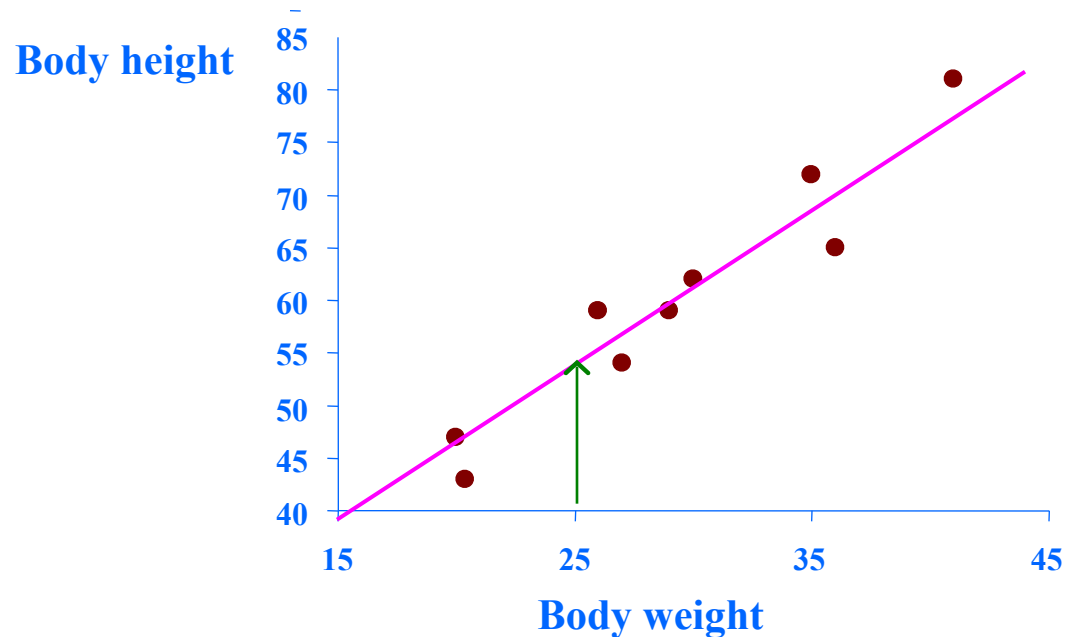
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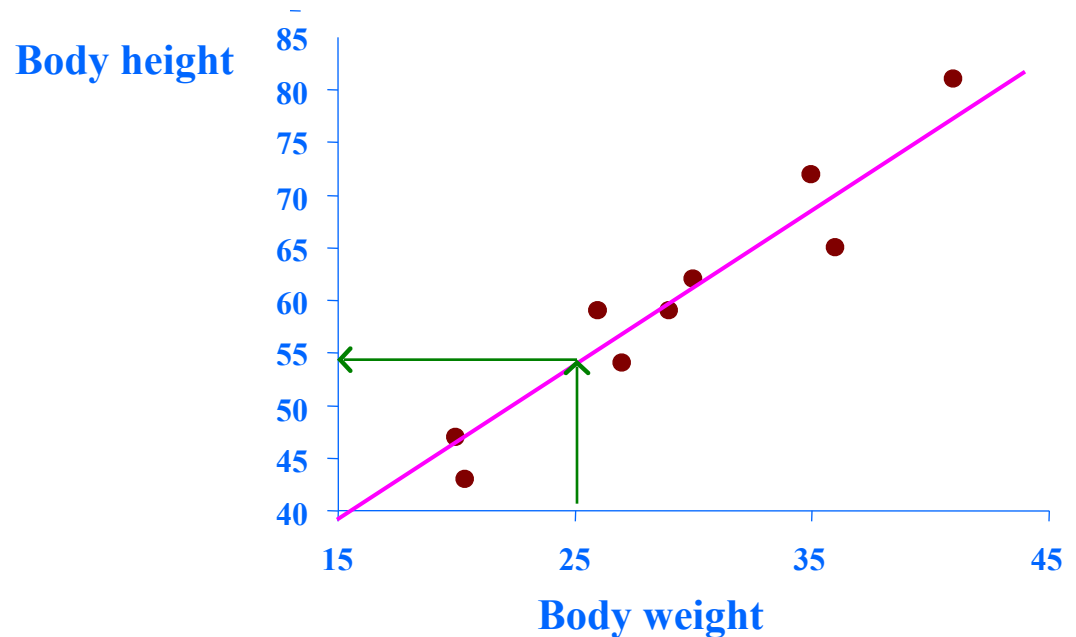
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Express what we **don't know** (= “dependent variable”)  
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# Linear Auto Regression

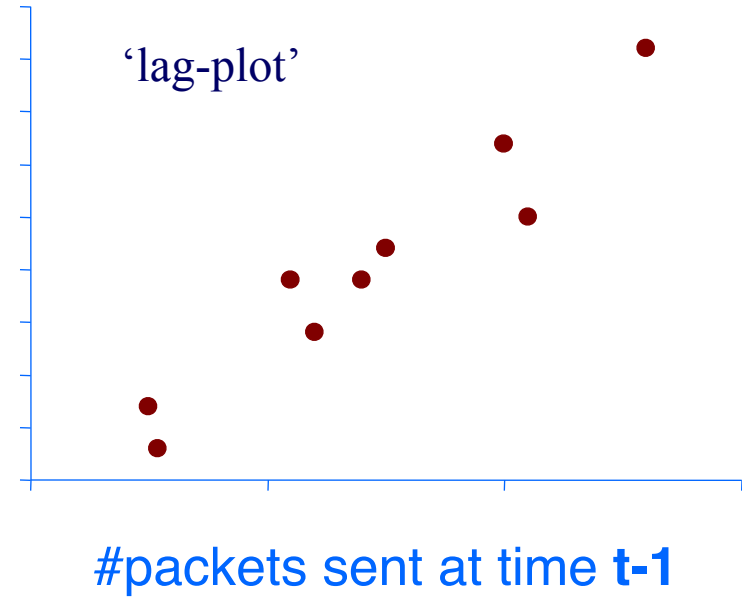
<i>Time</i>	<i>Packets Sent(t)</i>
1	43
2	54
3	72
...	...
N	??



# Linear Auto Regression

<i>Time</i>	<i>Packets Sent (t-1)</i>	<i>Packets Sent(t)</i>
1	-	43
2	43	54
3	54	72
...	...	...
N	25	??

#packets sent  
at time  $t$



**Lag  $w = 1$**

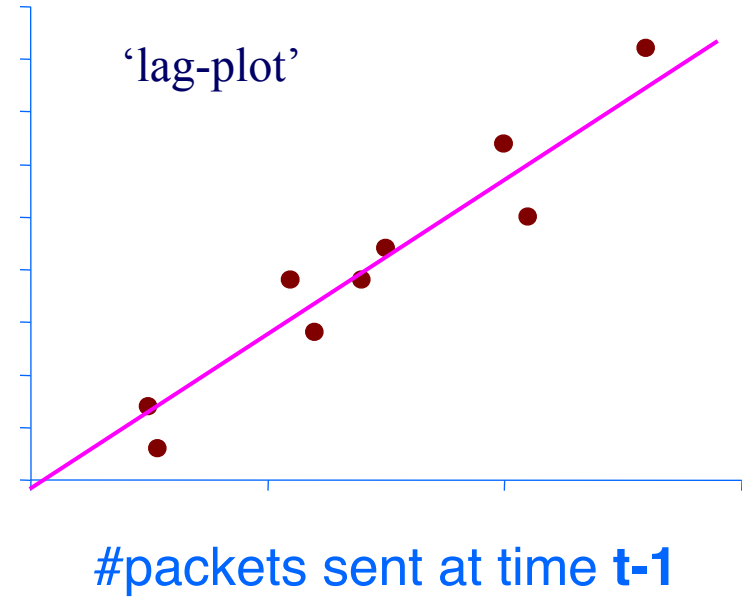
Dependent variable = # of packets sent ( $S[t]$ )

Independent variable = # of packets sent ( $S[t-1]$ )

# Linear Auto Regression

<i>Time</i>	<i>Packets Sent (t-1)</i>	<i>Packets Sent(t)</i>
1	-	43
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**Lag  $w = 1$**

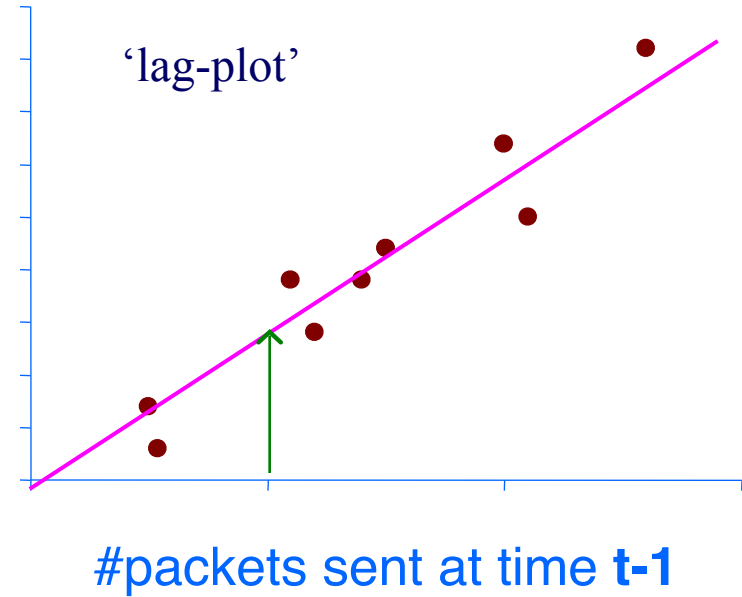
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<i>Time</i>	<i>Packets Sent (t-1)</i>	<i>Packets Sent(t)</i>
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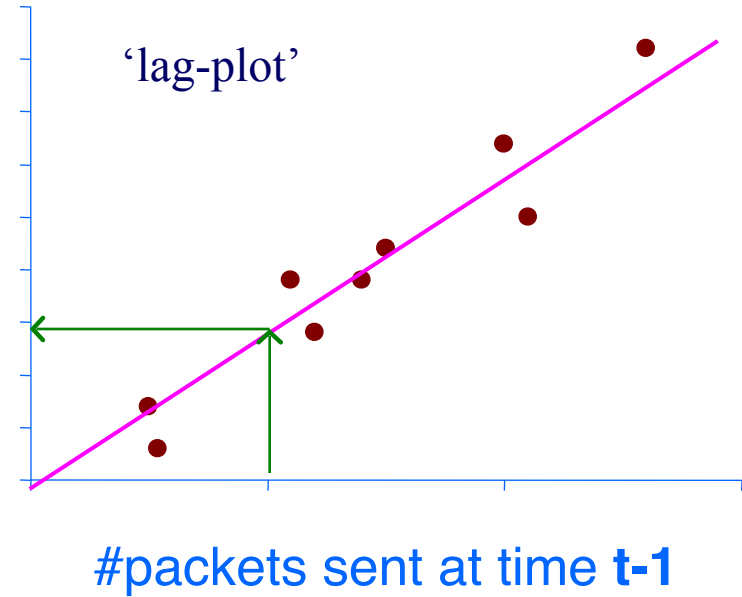
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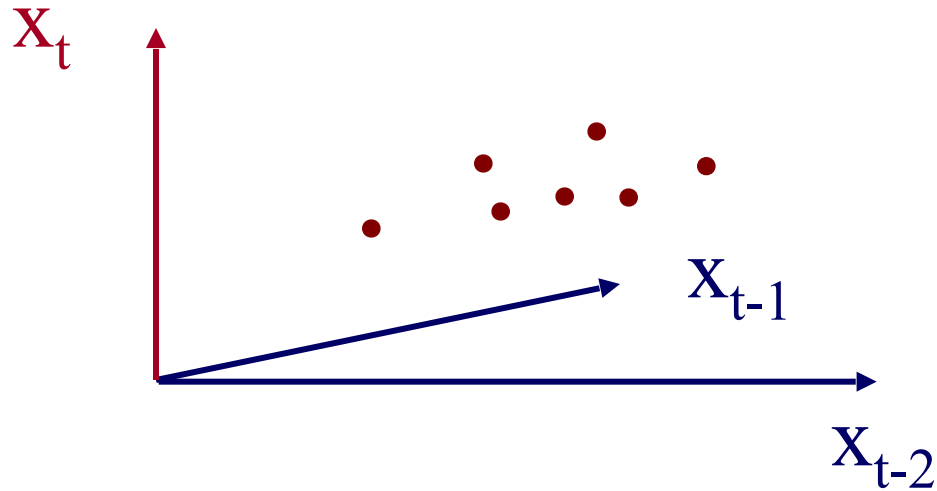
**Lag  $w = 1$**

Dependent variable = # of packets sent ( $S[t]$ )

Independent variable = # of packets sent ( $S[t-1]$ )

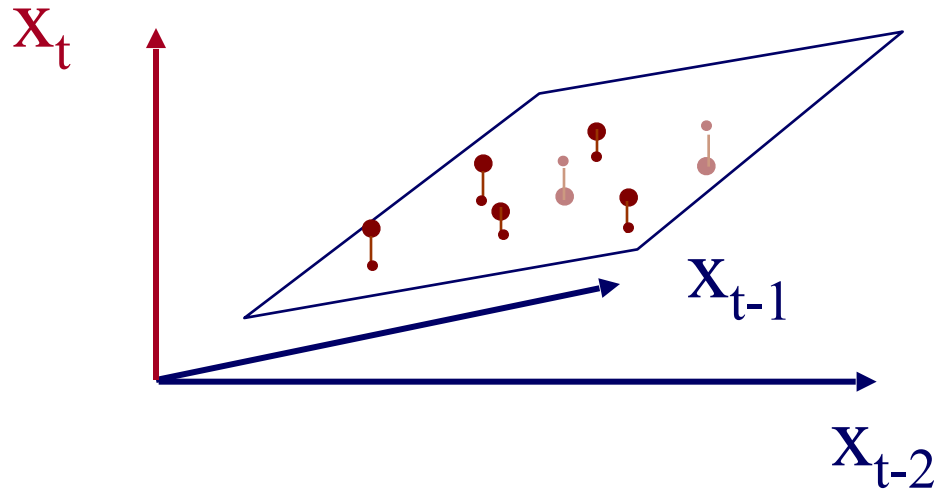
# More details:

- Q1: Can it work with window  $w > 1$ ?
- A1: YES!



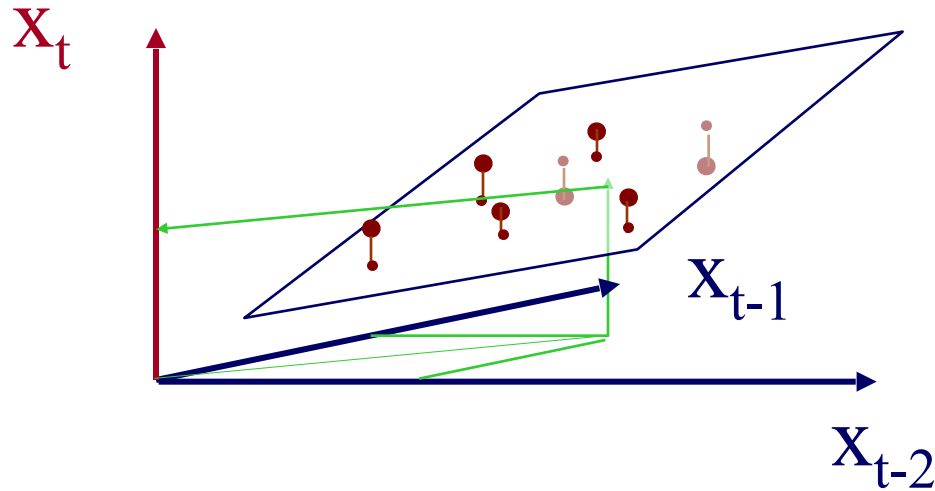
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- Q1: Can it work with window  $w > 1$ ?
- A1: YES! (we'll fit a hyper-plane, then!)



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# More details:

- Q1: Can it work with window  $w > 1$ ?
- A1: YES! The problem becomes:

$$\mathbf{X}_{[N \times w]} \times \mathbf{a}_{[w \times 1]} = \mathbf{y}_{[N \times 1]}$$

- OVER-CONSTRAINED
  - $\mathbf{a}$  is the vector of the regression coefficients
  - $\mathbf{X}$  has the  $N$  values of the  $w$  indep. variables
  - $\mathbf{y}$  has the  $N$  values of the dependent variable

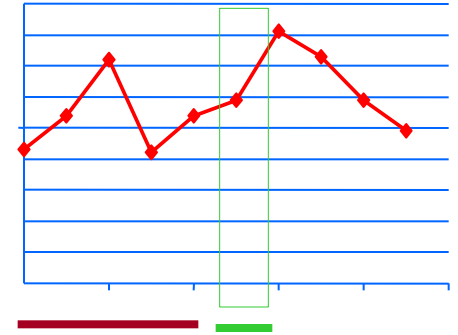


# More details:

- $\mathbf{X}_{[N \times w]} \times \mathbf{a}_{[w \times 1]} = \mathbf{y}_{[N \times 1]}$

Ind-var 1

Ind-var-w



time

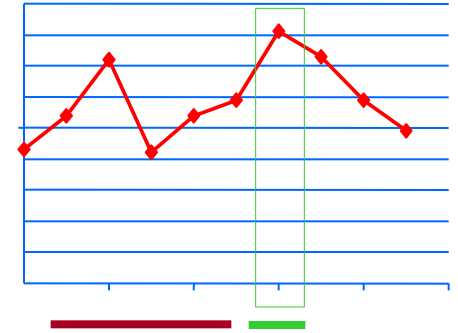
$$\begin{bmatrix}
 \underline{X_{11}, X_{12}, \dots, X_{1w}} \\
 X_{21}, X_{22}, \dots, X_{2w} \\
 \vdots \\
 \vdots \\
 \vdots \\
 X_{N1}, X_{N2}, \dots, X_{Nw}
 \end{bmatrix}
 \times
 \begin{bmatrix}
 a_1 \\
 a_2 \\
 \vdots \\
 a_w
 \end{bmatrix}
 =
 \begin{bmatrix}
 \underline{y_1} \\
 y_2 \\
 \vdots \\
 \vdots \\
 \vdots \\
 y_N
 \end{bmatrix}$$

# More details:

- $$\mathbf{X}_{[N \times w]} \times \mathbf{a}_{[w \times 1]} = \mathbf{y}_{[N \times 1]}$$

Ind-var 1

Ind-var-w



time

$$\begin{bmatrix} X_{11}, X_{12}, \dots, X_{1w} \\ X_{21}, X_{22}, \dots, X_{2w} \\ \vdots \\ \vdots \\ \vdots \\ X_{N1}, X_{N2}, \dots, X_{Nw} \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_w \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ y_N \end{bmatrix}$$

The diagram illustrates the matrix multiplication. The first matrix is labeled 'time' on the left with a downward arrow. The second matrix is labeled 'Ind-var-w' above it. The third matrix is labeled 'y' on the right. A red horizontal bar is under the second row of the first matrix, and a green horizontal bar is under the second element of the second matrix. A green vertical box is around the second element of the second matrix, and a red horizontal bar is below the x-axis of the graph above.

# More details

- Q2: How to estimate  $a_1, a_2, \dots, a_w = \mathbf{a}$ ?
- A2: with Least Squares fit

$$\mathbf{a} = (\mathbf{X}^T \times \mathbf{X})^{-1} \times (\mathbf{X}^T \times \mathbf{y})$$

- (Moore-Penrose pseudo-inverse)
- $\mathbf{a}$  is the vector that minimizes the RMSE from  $\mathbf{y}$

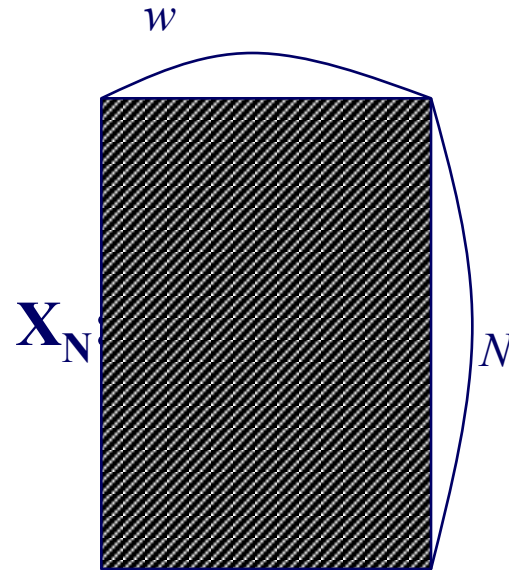
# More details

- Straightforward solution:

$$\mathbf{a} = (\mathbf{X}^T \times \mathbf{X})^{-1} \times (\mathbf{X}^T \times \mathbf{y})$$

$\mathbf{a}$  : Regression Coeff. Vector

$\mathbf{X}$  : Sample Matrix



- Observations:

- Sample matrix  $\mathbf{X}$  grows over time
- needs matrix inversion
- $\mathbf{O}(N \times w^2)$  computation
- $\mathbf{O}(N \times w)$  storage

# Even more details

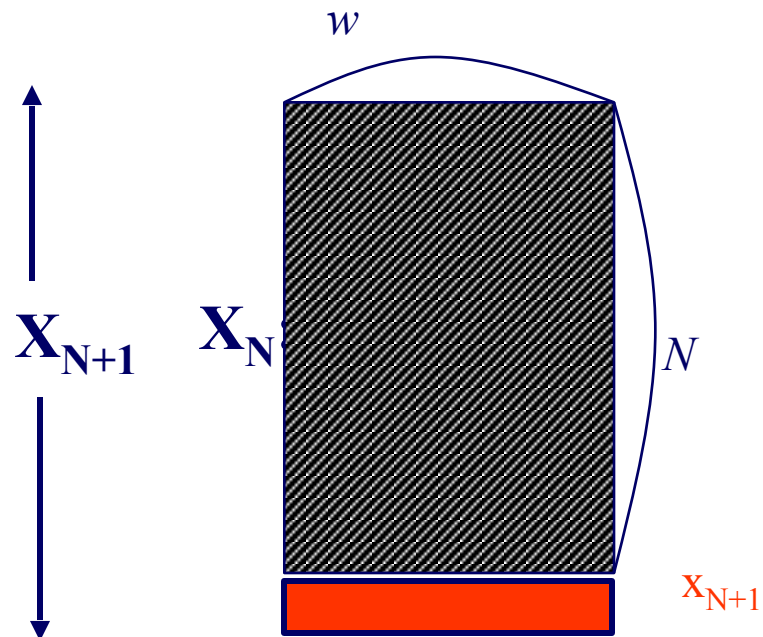
- Q3: Can we estimate  $\mathbf{a}$  incrementally?
- A3: Yes, with the brilliant, classic method of “Recursive Least Squares” (RLS) (see, e.g., [Yi+00], for details).
- We can do the matrix inversion, **WITHOUT** inversion! (How is that possible?!)

# Even more details

- Q3: Can we estimate  $\mathbf{a}$  incrementally?
- A3: Yes, with the brilliant, classic method of **“Recursive Least Squares” (RLS)** (see, e.g., [Yi+00], for details).
- We can do the matrix inversion, **WITHOUT** inversion! (How is that possible?!)
- A: our matrix has special form:  $(\mathbf{X}^T \mathbf{X})$

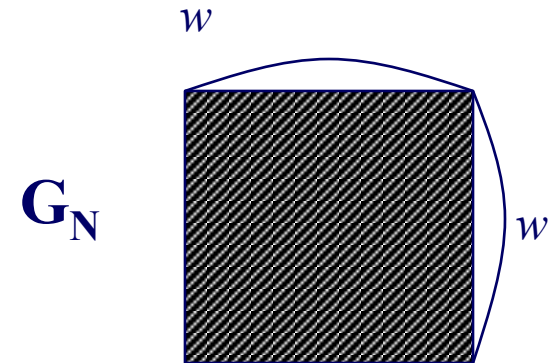
# More details

At the  $N+1$  time tick:



# More details: key ideas

- Let  $\mathbf{G}_N = (\mathbf{X}_N^T \times \mathbf{X}_N)^{-1}$  (“gain matrix”)
- $\mathbf{G}_{N+1}$  can be computed recursively from  $\mathbf{G}_N$  without matrix inversion





# Comparison:

- **Straightforward Least Squares**

- Needs huge matrix (**growing** in size)  
 $O(N \times w)$
- Costly matrix operation  
 $O(N \times w^2)$

- **Recursive LS**

- Need much smaller, fixed size matrix  
 $O(w \times w)$
- Fast, incremental computation  
 $O(1 \times w^2)$
- **no matrix inversion**

$$N = 10^6, \quad w = 1-100$$

## EVEN more details:

$$G_{N+1} = G_N - [c]^{-1} \times [G_N \times x_{N+1}^T] \times x_{N+1} \times G_N$$

$1 \times w$  row vector



$$c = [1 + x_{N+1} \times G_N \times x_{N+1}^T]$$

Let's elaborate  
(VERY IMPORTANT, VERY VALUABLE!)

## **EVEN more details:**

$$a = [X_{N+1}^T \times X_{N+1}]^{-1} \times [X_{N+1}^T \times y_{N+1}]$$

# EVEN more details:

$$a = [X_{N+1}^T \times X_{N+1}]^{-1} \times [X_{N+1}^T \times y_{N+1}]$$

$[w \times 1]$

$[w \times (N+1)]$

$[(N+1) \times w]$

$[w \times (N+1)]$

$[(N+1) \times 1]$

## EVEN more details:

$$a = \left[ X_{N+1}^T \times X_{N+1} \right]^{-1} \times \left[ X_{N+1}^T \times y_{N+1} \right]$$

$[w \times (N+1)]$        $[(N+1) \times w]$

## EVEN more details:

$$a = [X_{N+1}^T \times X_{N+1}]^{-1} \times [X_{N+1}^T \times y_{N+1}]$$

‘gain  
matrix’

$$G_{N+1} \equiv [X_{N+1}^T \times X_{N+1}]^{-1}$$

$$G_{N+1} = G_N - [c]^{-1} \times [G_N \times x_{N+1}^T] \times x_{N+1} \times G_N$$

wxw
wxw
1x1
wxw
wx1
1xw
wxw

**SCALAR!**  $c = [1 + x_{N+1} \times G_N \times x_{N+1}^T]$

# Altogether:

$$G_0 \equiv \delta I$$

where

$I$ :  $w \times w$  identity matrix

$\delta$ : a large positive number

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- **Straightforward Least Squares**

- Needs huge matrix (**growing** in size)  
 $O(N \times w)$
- Costly matrix operation  
 $O(N \times w^2)$

- **Recursive LS**

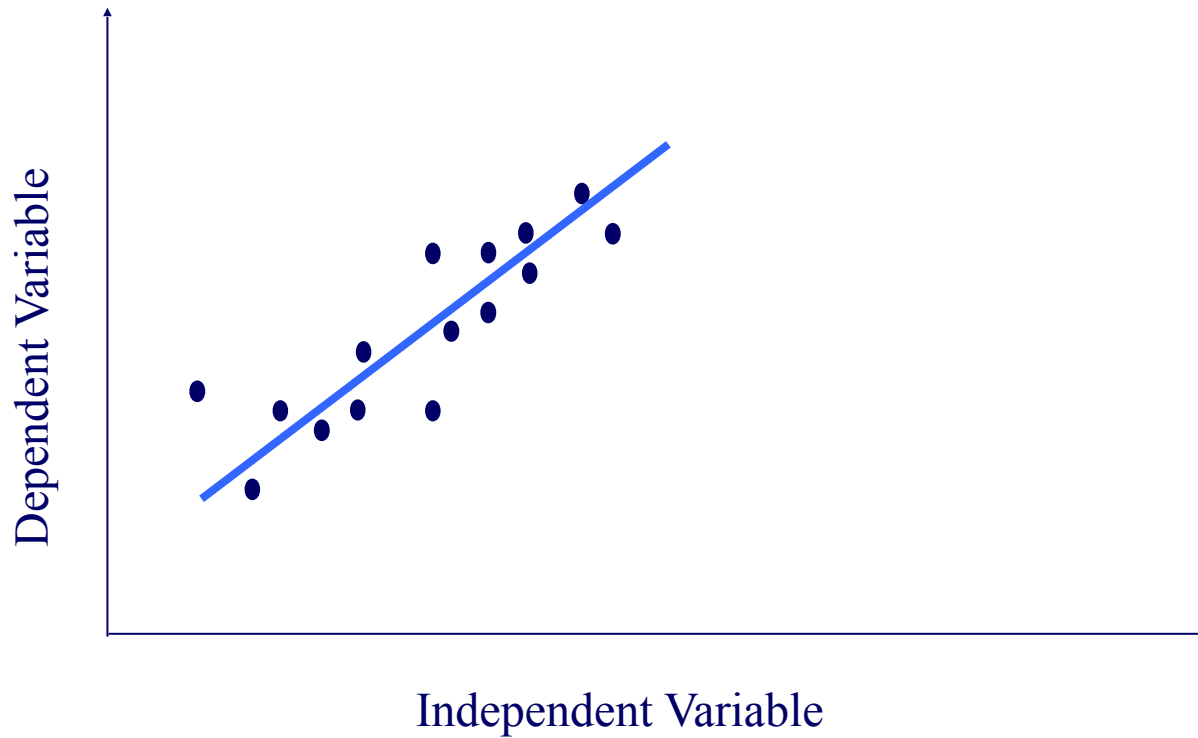
- Need much smaller, fixed size matrix  
 $O(w \times w)$
- Fast, incremental computation  
 $O(1 \times w^2)$
- **no matrix inversion**

$$N = 10^6, \quad w = 1-100$$

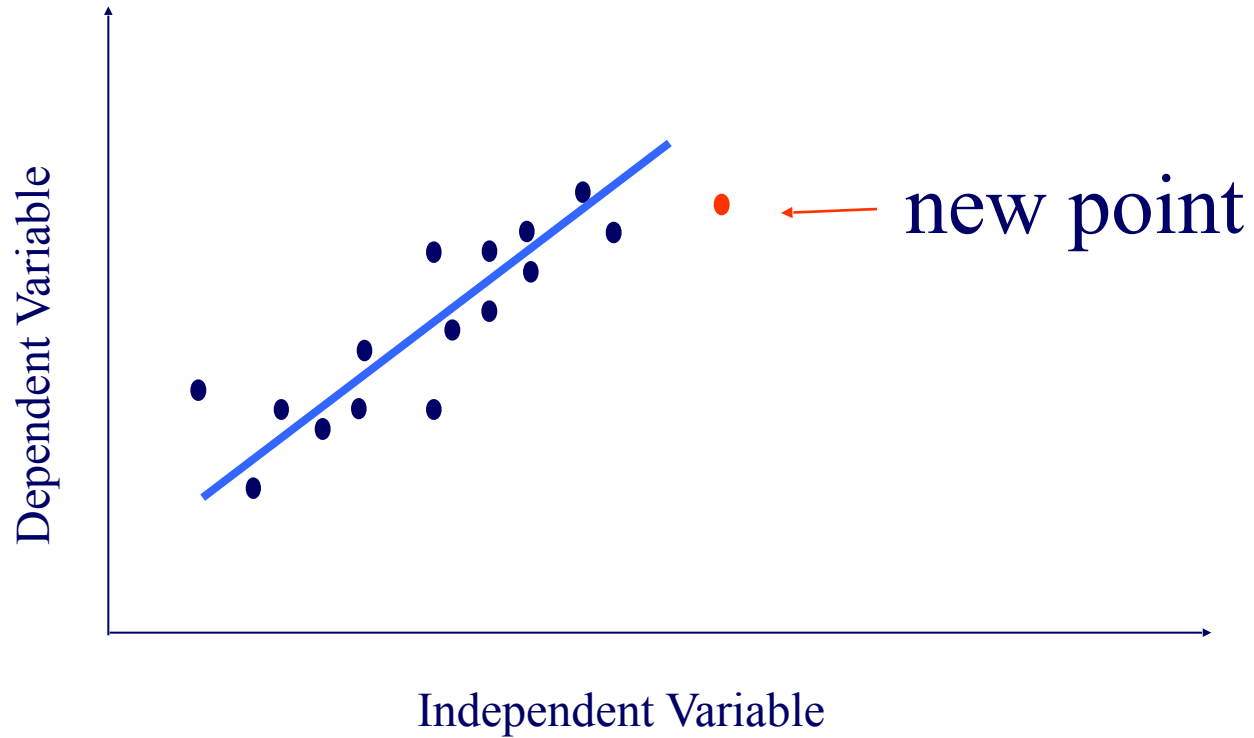


# Pictorially:

- Given:

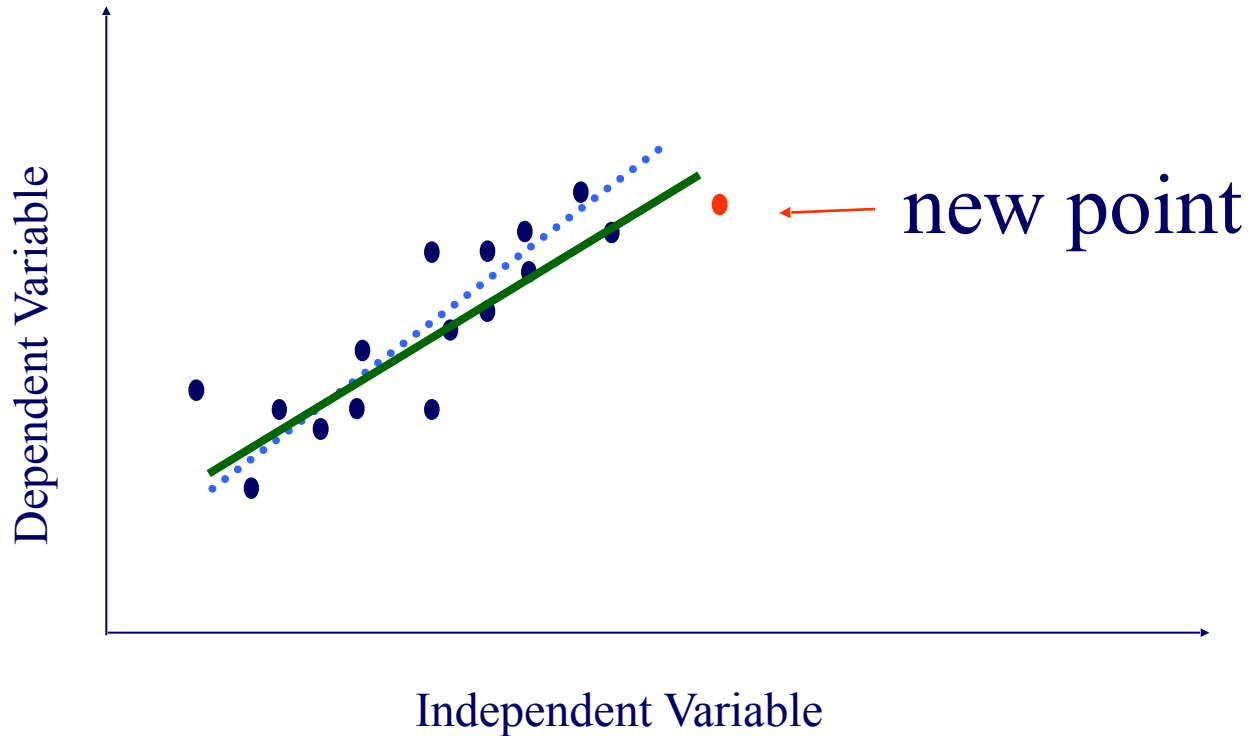


# Pictorially:



# Pictorially:

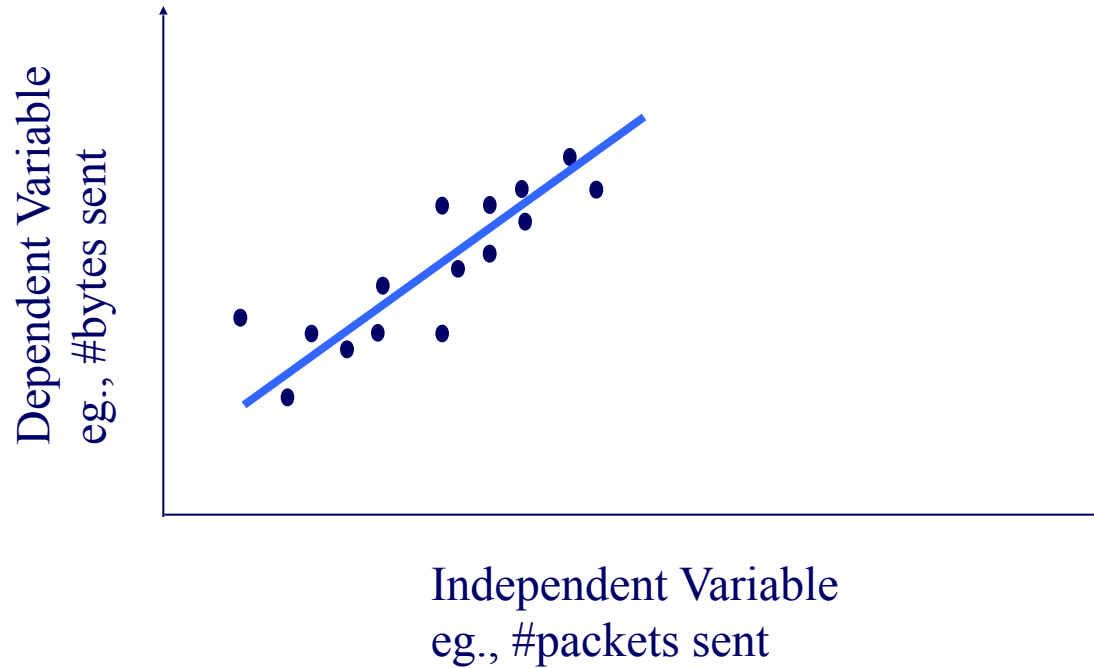
RLS: quickly compute new best fit



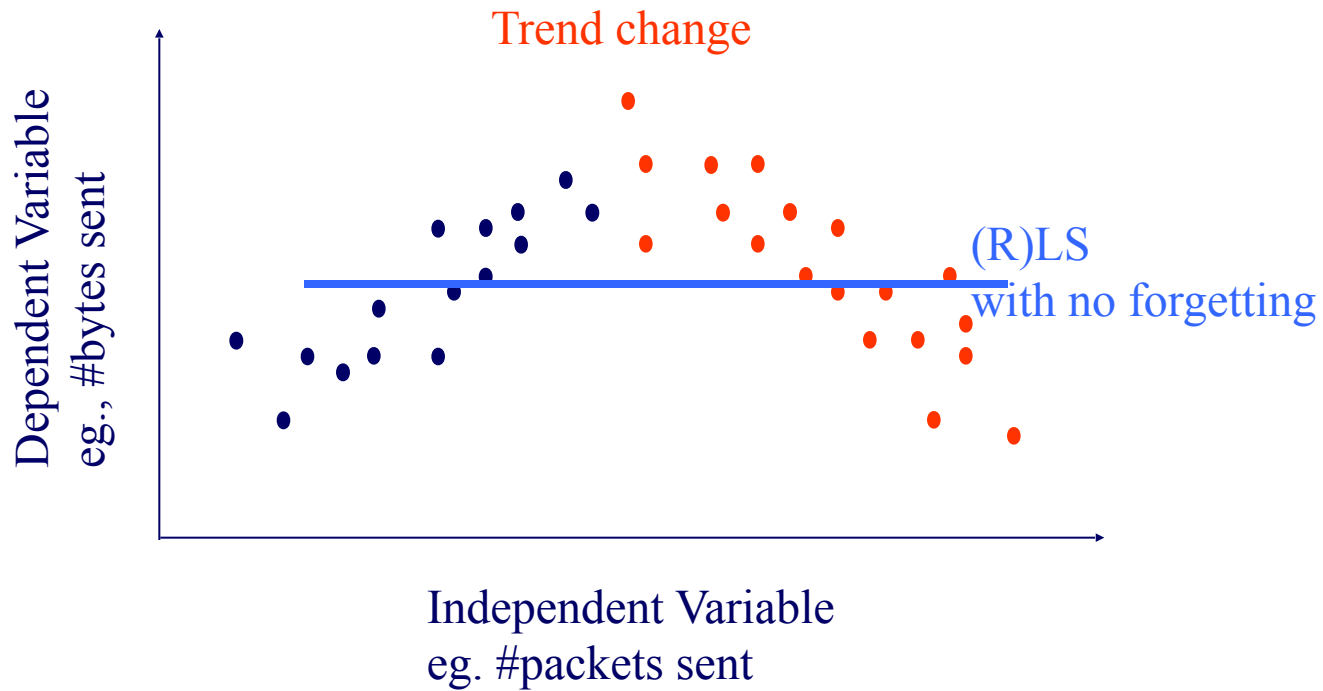
# Even more details

- Q4: can we ‘forget’ the older samples?
- A4: Yes - RLS can easily handle that  $[Y_{i+00}]$ :

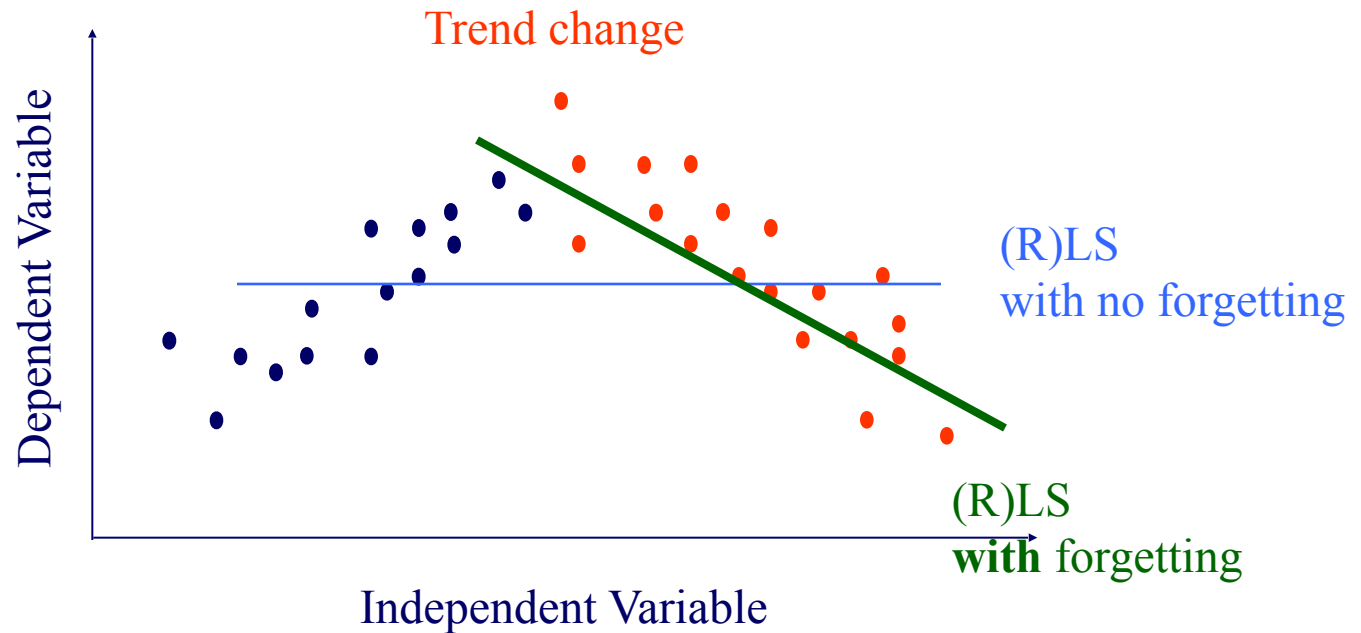
# Adaptability - 'forgetting'



# Adaptability - 'forgetting'



# Adaptability - 'forgetting'



- RLS: can \*trivially\* handle 'forgetting'