CSE 6242 / CX 4242
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# Time Series

## Mining and Forecasting

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Slides based on Prof. Christos Faloutsos's materials

# Outline

- Motivation
- Similarity search – distance functions
- Linear Forecasting
- Non-linear forecasting
- Conclusions

# Problem definition

- **Given**: one or more sequences

  $x_1, x_2, \ldots, x_t, \ldots$

  $(y_1, y_2, \ldots, y_t, \ldots)$

  $(\ldots)$
- **Find**
  - similar sequences; forecasts
  - patterns; clusters; outliers

# Motivation - Applications

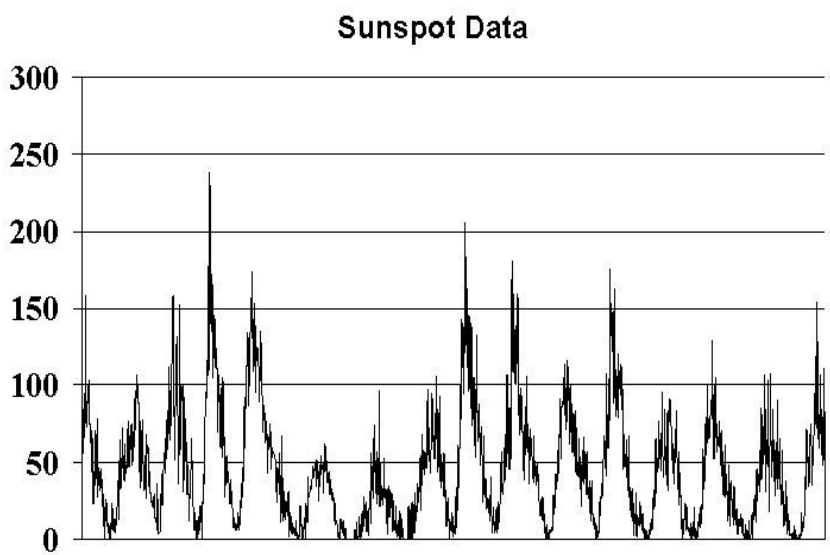
- Financial, sales, economic series
- Medical
  - ECGs +; blood pressure etc monitoring
  - reactions to new drugs
  - elderly care

# Motivation - Applications (cont'd)

- 'Smart house'
  - sensors monitor temperature, humidity, air quality
- video surveillance

# Motivation - Applications (cont'd)

- Weather, environment/anti-pollution
  - volcano monitoring
  - air/water pollutant monitoring

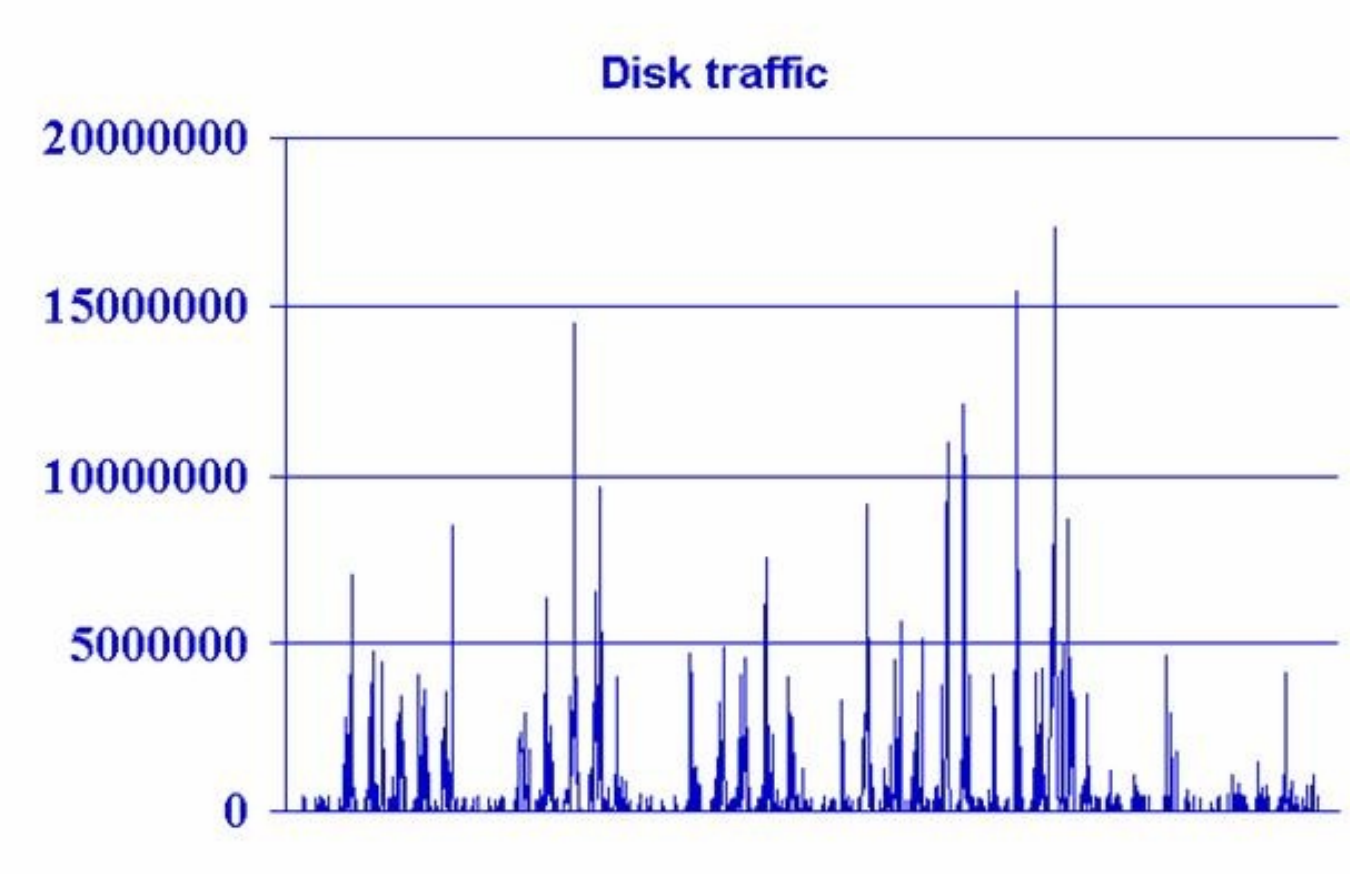


# Motivation - Applications (cont'd)

- Computer systems
  - 'Active Disks' (buffering, prefetching)
  - web servers (ditto)
  - network traffic monitoring
  - ...

# Stream Data: Disk accesses

#bytes



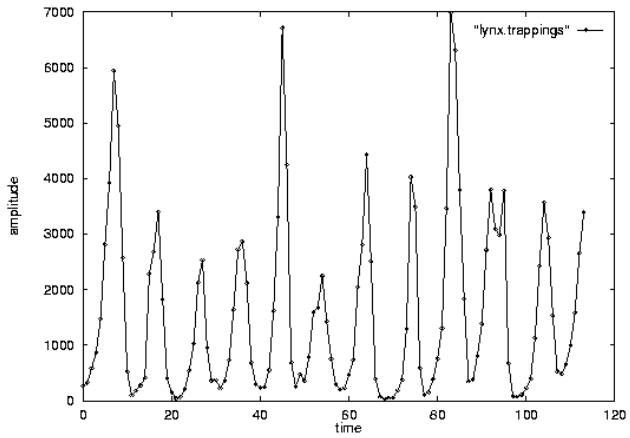

time

# Problem #1:

**Goal:** given a signal (e.g.., #packets over time)

**Find:** patterns, periodicities, and/or compress

count

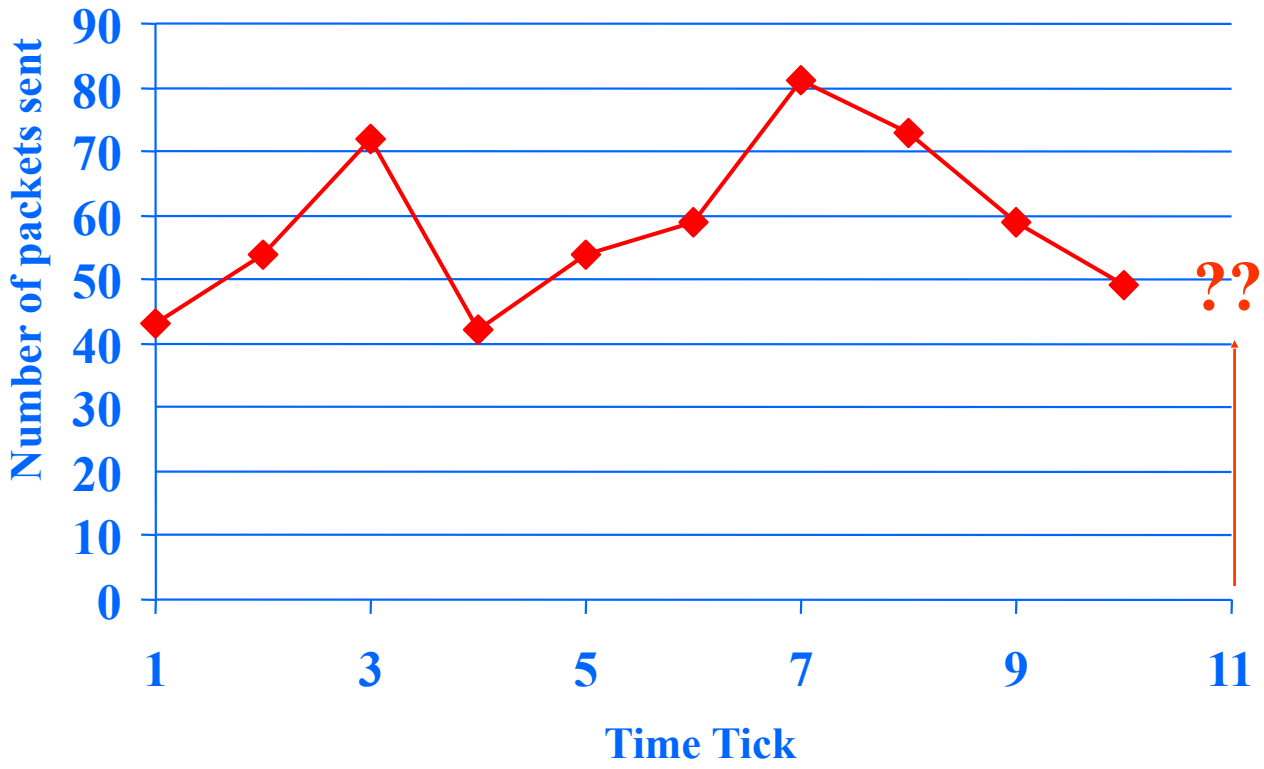



lynx caught per year
(packets per day;
temperature per day)

year

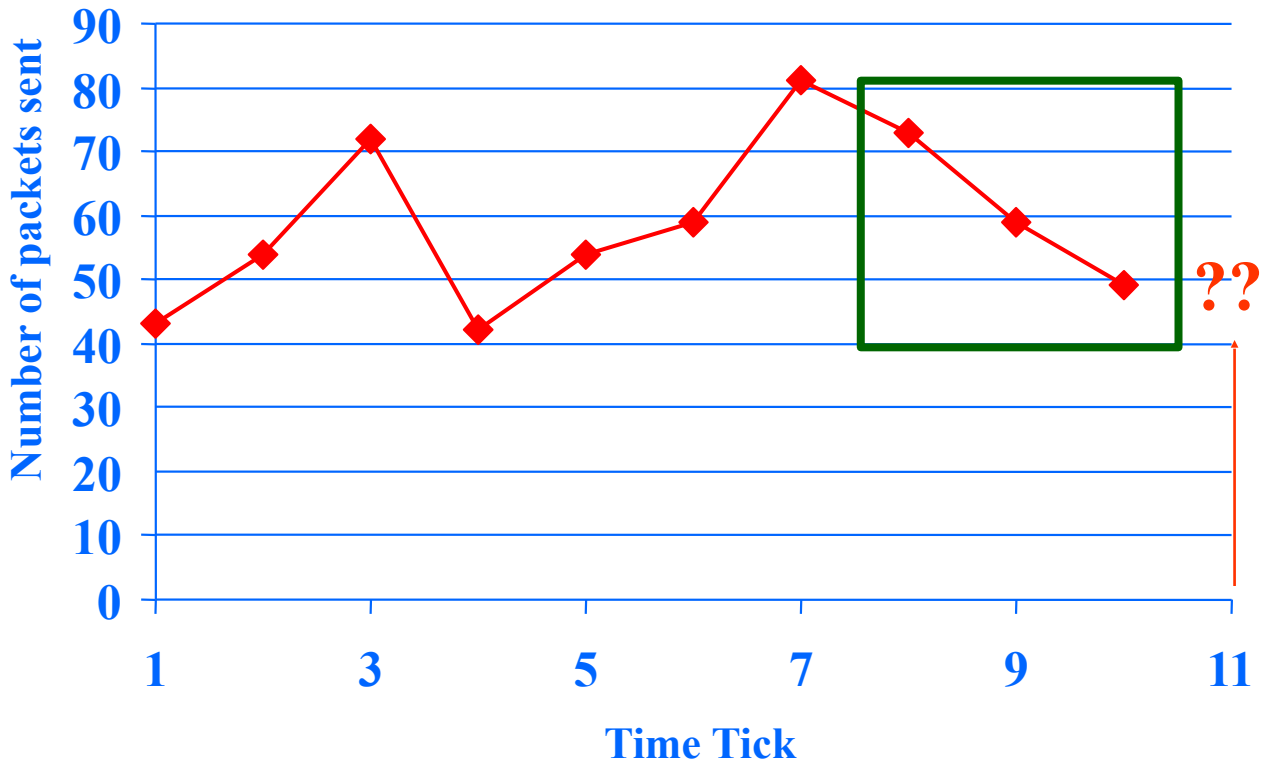
# Problem#2: Forecast

Given $x_t$, $x_{t-1}$, …, forecast $x_{t+1}$



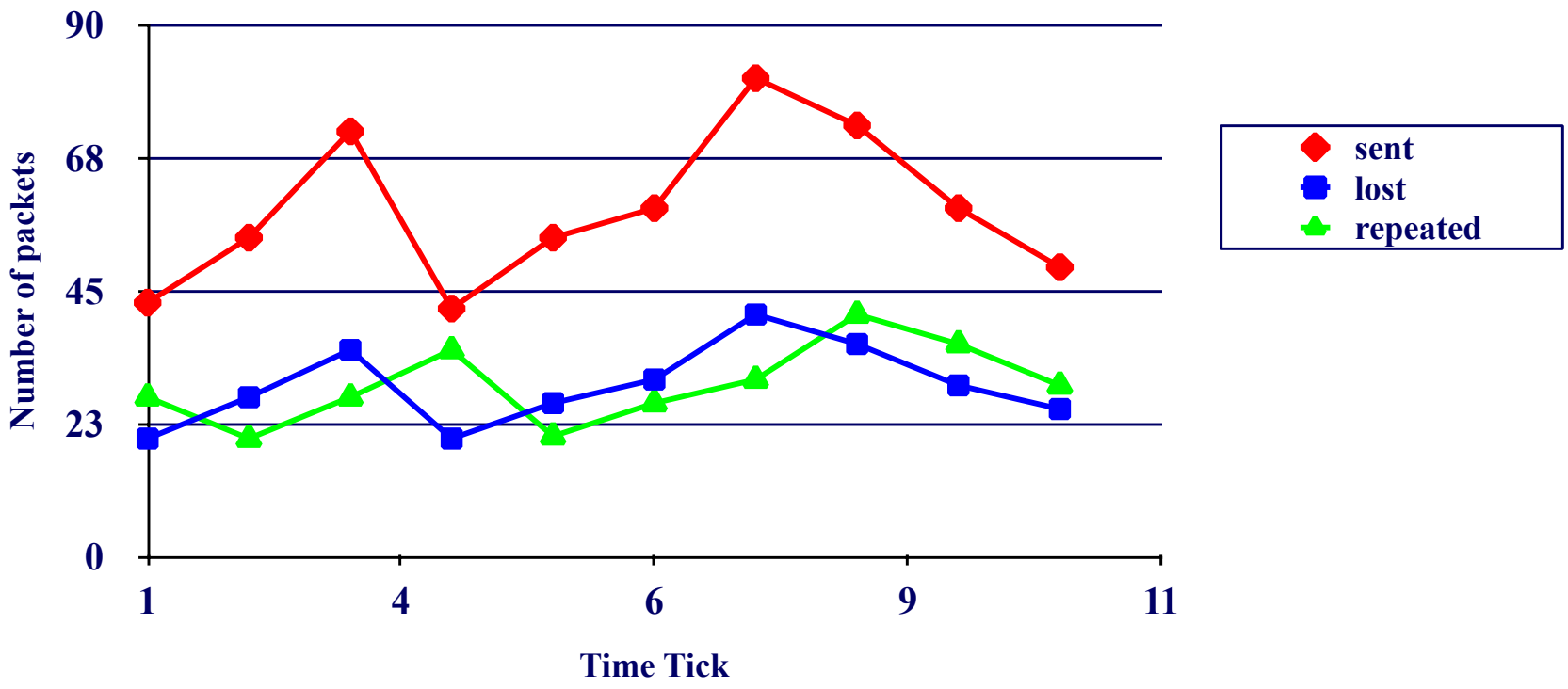
# Problem#2': Similarity search

E.g.., Find a 3-tick pattern, similar to the last one



# Problem #3:

- Given: A set of **correlated** time sequences
- Forecast 'Sent(t)'



# Important observations

Patterns, rules, forecasting and similarity indexing are closely related:

- To do forecasting, we need
  - to find patterns/rules
  - to find similar settings in the past
- to find outliers, we need to have forecasts
  - (outlier = too far away from our forecast)

# Outline

- Motivation
- Similarity Search and Indexing
- Linear Forecasting
- Non-linear forecasting
- Conclusions

# Outline

- Motivation
- Similarity search and distance functions
  - Euclidean
  - Time-warping
- ...

# Importance of distance functions

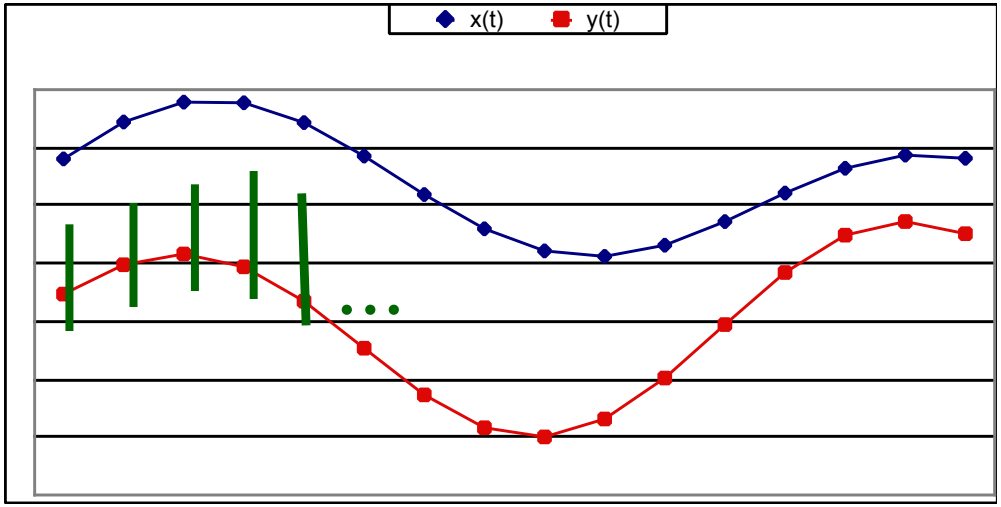
Subtle, but **absolutely necessary**:

- A ‘must’ for similarity indexing (-> forecasting)
- A ‘must’ for clustering

Two major families

- Euclidean and Lp norms
- Time warping and variations

# Euclidean and Lp

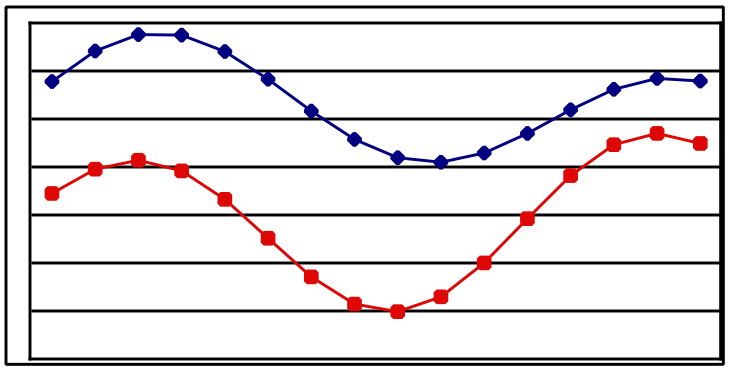



$$D(\vec{x},\vec{y}) = \sum_{i=1}^{n} (x_i - y_i)^2$$

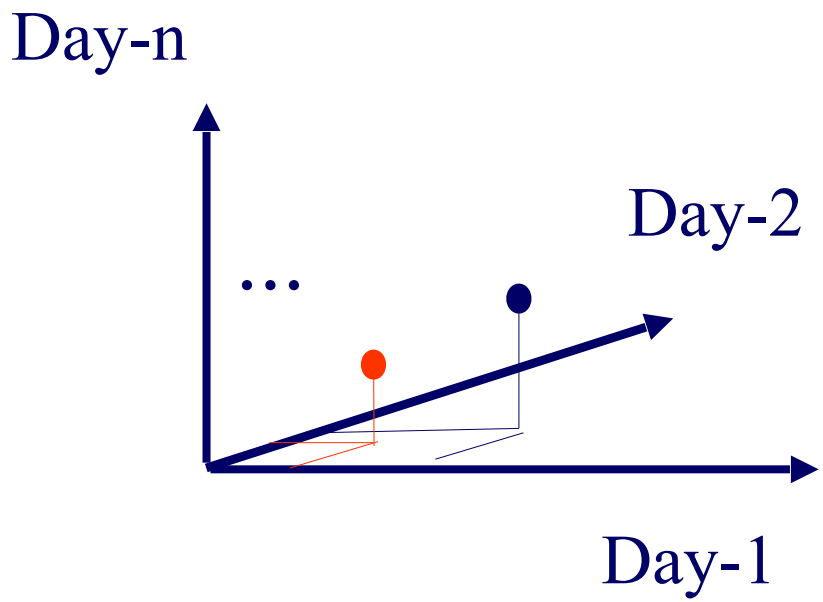
$$L_p(\vec{x},\vec{y}) = \sum_{i=1}^{n} |x_i - y_i|^p$$

- $L_1$: city-block = Manhattan
- $L_2$ = Euclidean
- $L_\infty$

# Observation #1



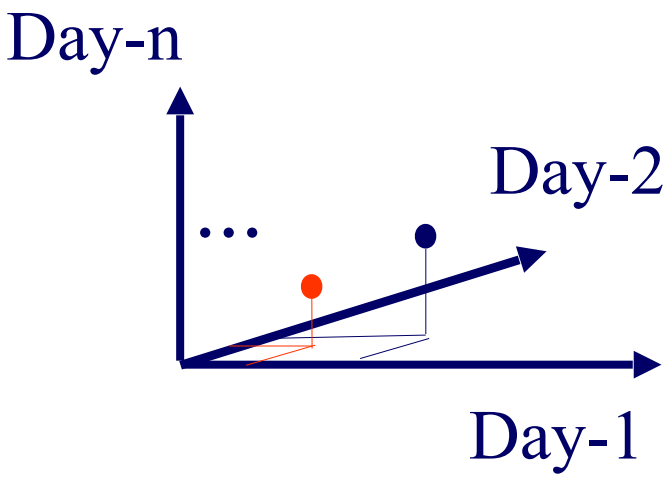
- **Time sequence -> n-d vector**

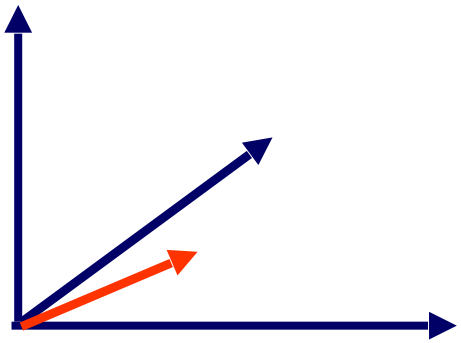


# Observation #2

Euclidean distance is closely related to

- cosine similarity
- dot product
- ‘cross-correlation’ function

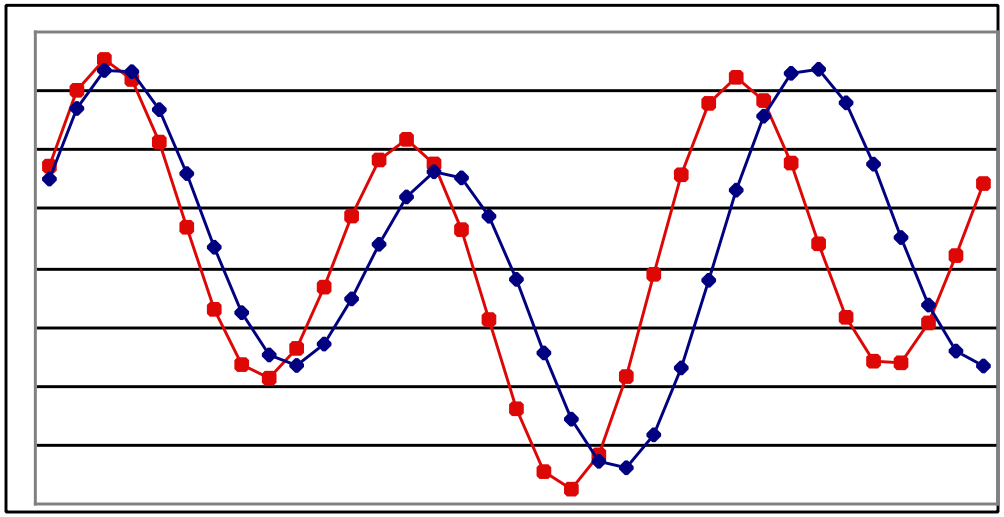


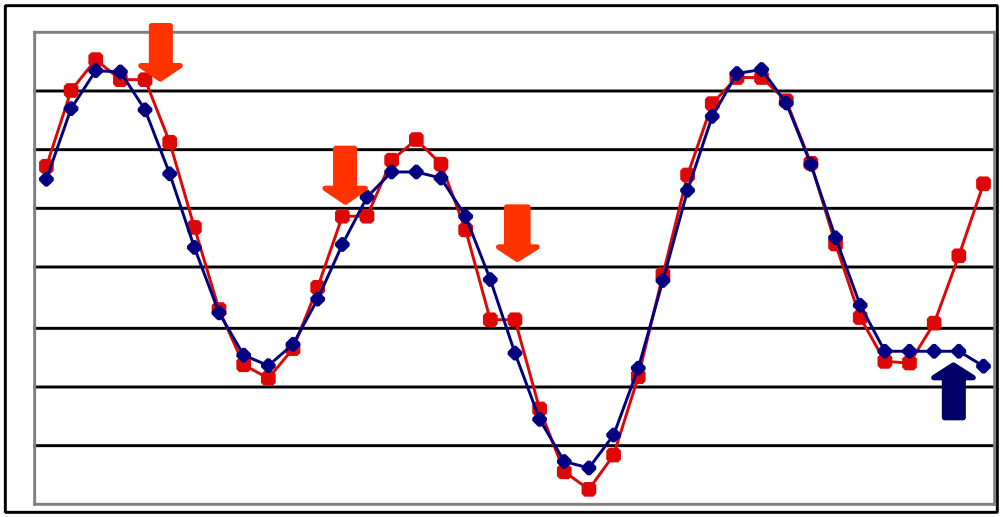
# Time Warping

- allow accelerations - decelerations
  - (with or w/o penalty)
- THEN compute the (Euclidean) distance (+ penalty)
- related to the string-editing distance

# Time Warping



‘stutters’:



# Time warping

Q: how to compute it?

A: dynamic programming

$D(i, j)$ = cost to match

prefix of length $i$ of first sequence $x$ with prefix of length $j$ of second sequence $y$

# Time warping

Thus, with no penalty for stutter, for sequences

$$x_1, x_2, \ldots, x_{i,;} \qquad y_1, y_2, \ldots, y_j$$

$$D(i,j) = \left\| x[i] - y[j] \right\| + \min \begin{cases} D(i-1, j-1) & \text{no stutter} \\ D(i, j-1) & \text{x-stutter} \\ D(i-1, j) & \text{y-stutter} \end{cases}$$

# Time warping

VERY SIMILAR to the string-editing distance

$$D(i,j) = \left\| x[i] - y[j] \right\| + \min \begin{cases} D(i-1, j-1) & \text{no stutter} \\ D(i, j-1) & \text{x-stutter} \\ D(i-1, j) & \text{y-stutter} \end{cases}$$

# Time warping

- Complexity: O(M*N) - quadratic on the length of the strings
- **<u>Many</u>** variations (penalty for stutters; limit on the number/percentage of stutters; …)
- popular in voice processing [Rabiner + Juang]

# Other Distance functions

- piece-wise linear/flat approx.; compare pieces [Keogh+01] [Faloutsos+97]
- ‘cepstrum’ (for voice [Rabiner+Juang])
  - do DFT; take log of amplitude; do DFT again!
- Allow for small gaps [Agrawal+95]

See tutorial by [Gunopulos + Das, SIGMOD01]

# Other Distance functions

- In [Keogh+, KDD’04]: parameter-free, MDL based

# Conclusions

Prevailing distances:

- Euclidean and
- time-warping

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# Linear Forecasting

# Forecasting

"Prediction is very difficult, especially about the future."

- Nils Bohr

Danish physicist and Nobel Prize laureate

# Outline

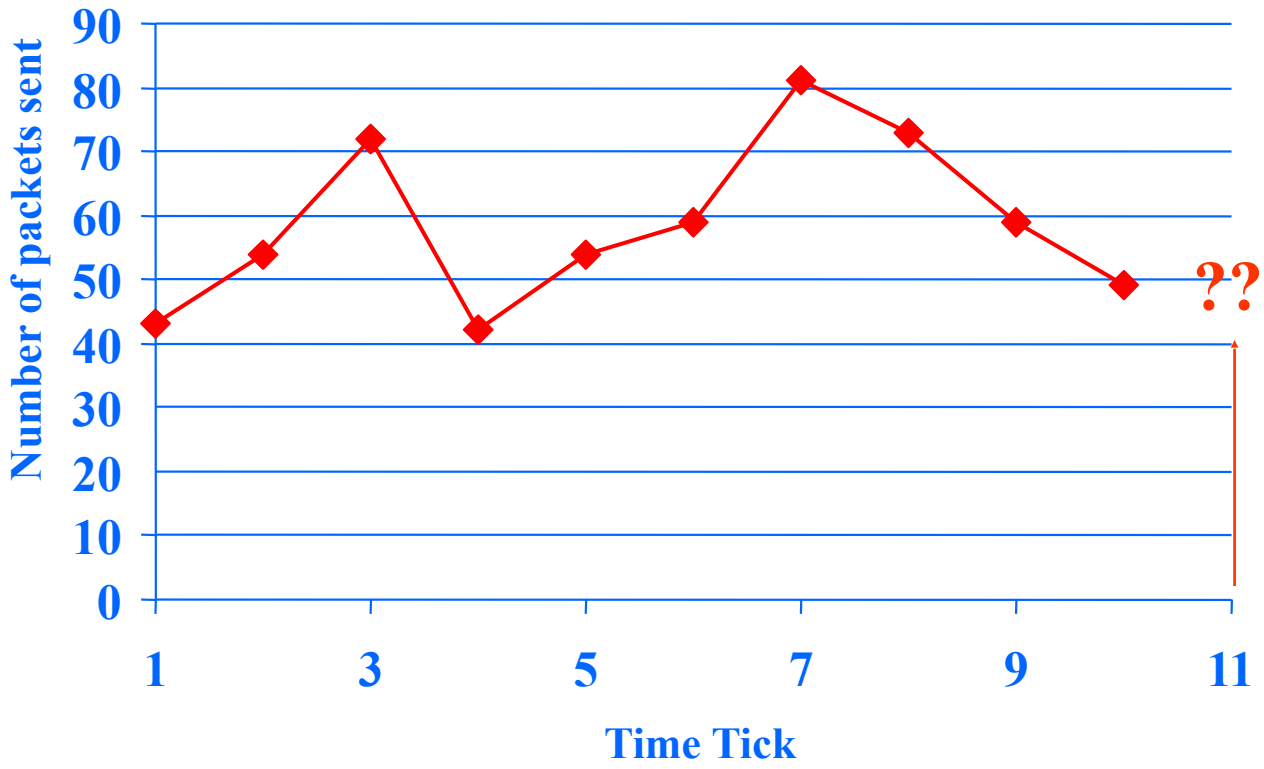
- Motivation
- ...
- Linear Forecasting
  - Auto-regression: Least Squares; RLS
  - Co-evolving time sequences
  - Examples
  - Conclusions

# Reference

[Yi+00] Byoung-Kee Yi et al.: *Online Data Mining for Co-Evolving Time Sequences*, ICDE 2000. (Describes MUSCLES and Recursive Least Squares)

# Problem#2: Forecast

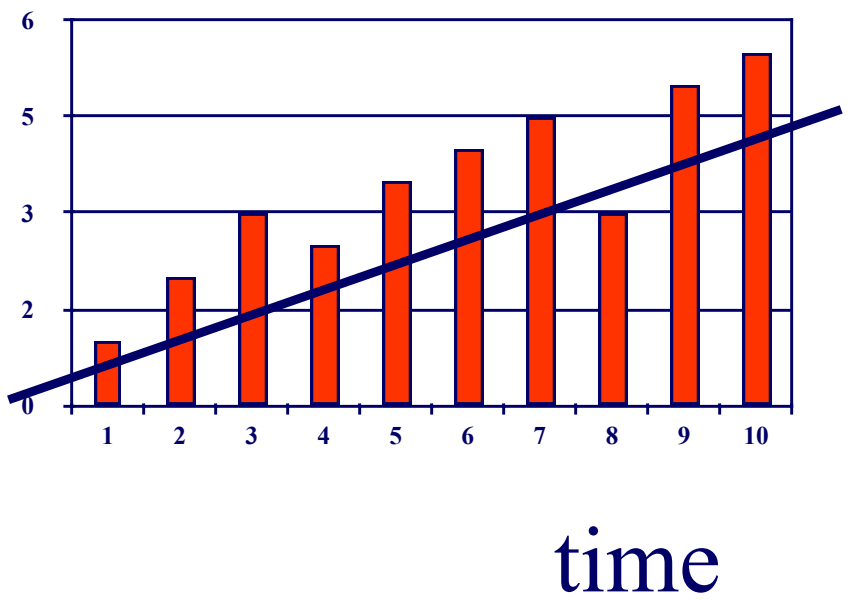
- Example: give $x_{t-1}$, $x_{t-2}$, …, forecast $x_t$



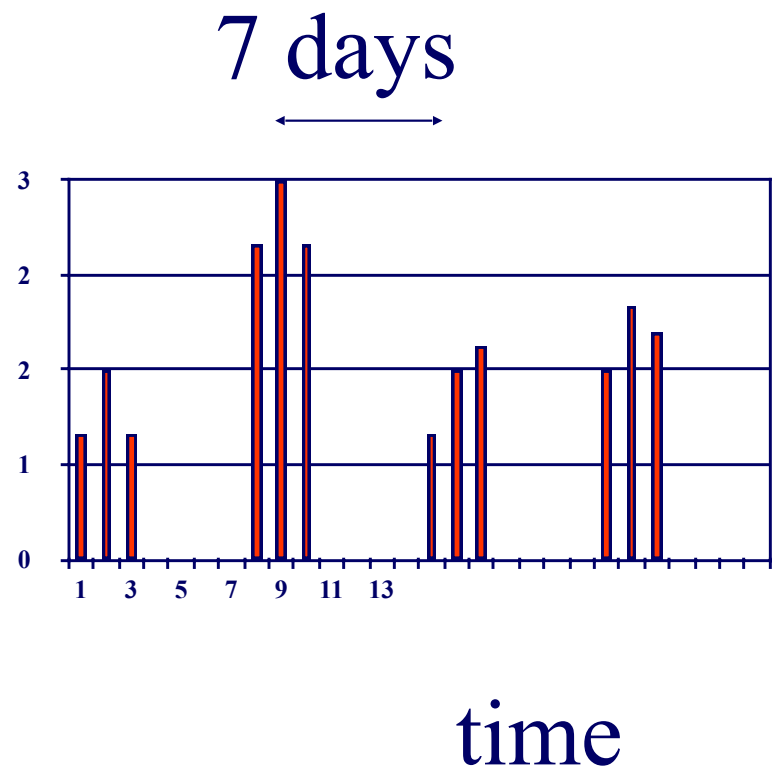
# Forecasting: Preprocessing

MANUALLY:

remove trends



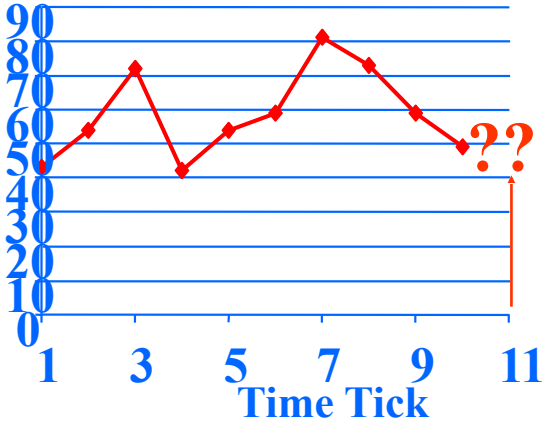

spot periodicities



# Problem#2: Forecast

- Solution: try to express
  $x_t$
  as a linear function of the past: $x_{t-1}$, $x_{t-2}$, …,
  (up to a window of $w$)

$$x_t \approx a_1 x_{t-1} + \ldots + a_w x_{t-w} + noise$$



# (Problem: Back-cast; interpolate)

- Solution - interpolate: try to express

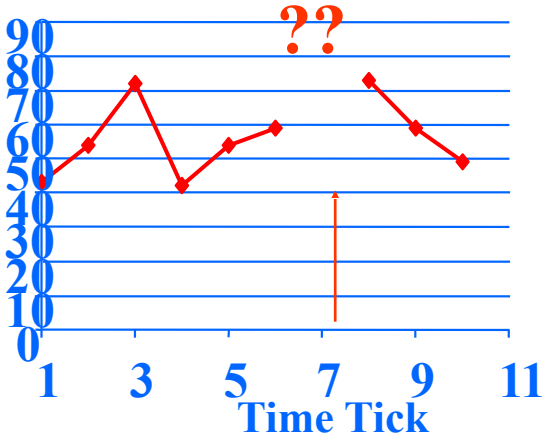
  $x_t$

  as a linear function of the past AND the future:

  $x_{t+1}, x_{t+2}, \ldots x_{t+wfuture;}\ x_{t-1}, \ldots x_{t-wpast}$

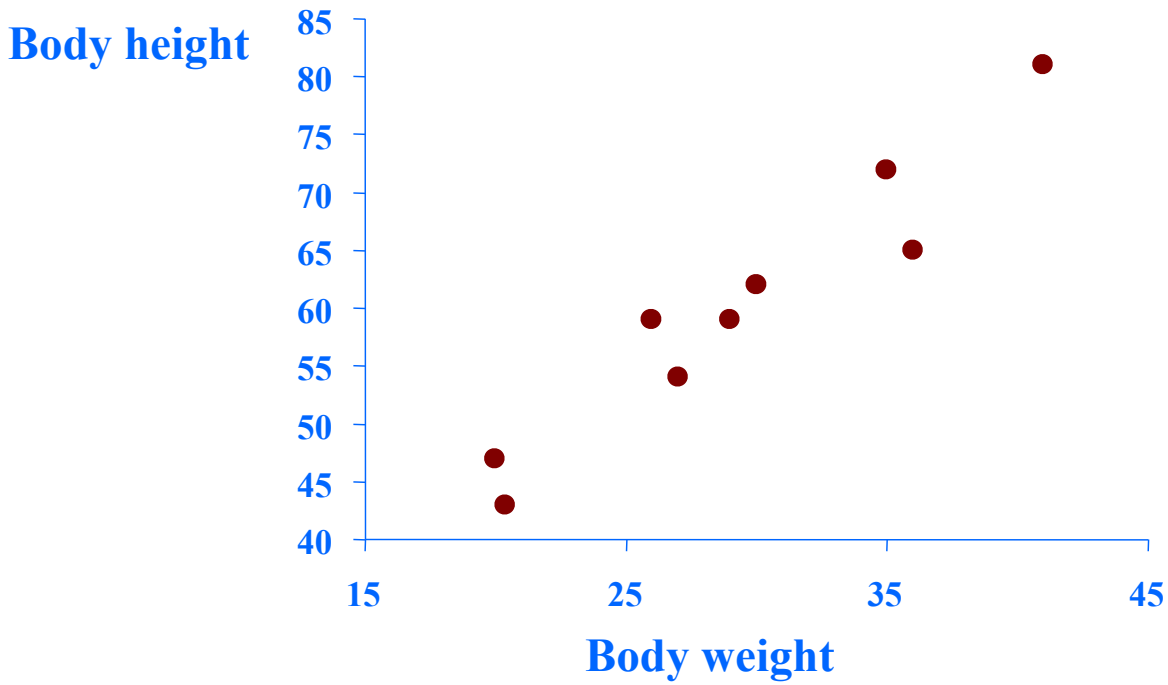
  (up to windows of $w_{past}$ , $w_{future}$)

- EXACTLY the same algo's



# Linear Regression: idea

| *patient* | *weight* | *height* |
|---|---|---|
| 1 | 27 | 43 |
| 2 | 43 | 54 |
| 3 | 54 | 72 |
| … | … | … |
| N | **25** | **??** |




- express what we **don't know** (= "dependent variable")
- as a linear function of what we **know** (= "independent variable(s)")

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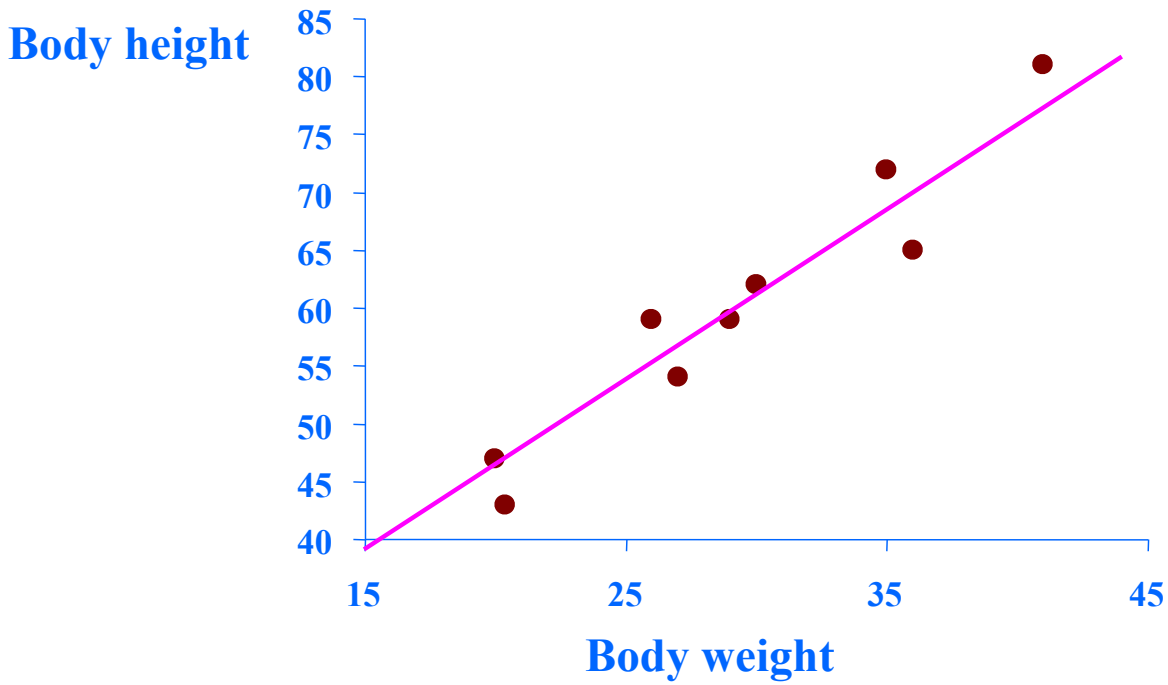


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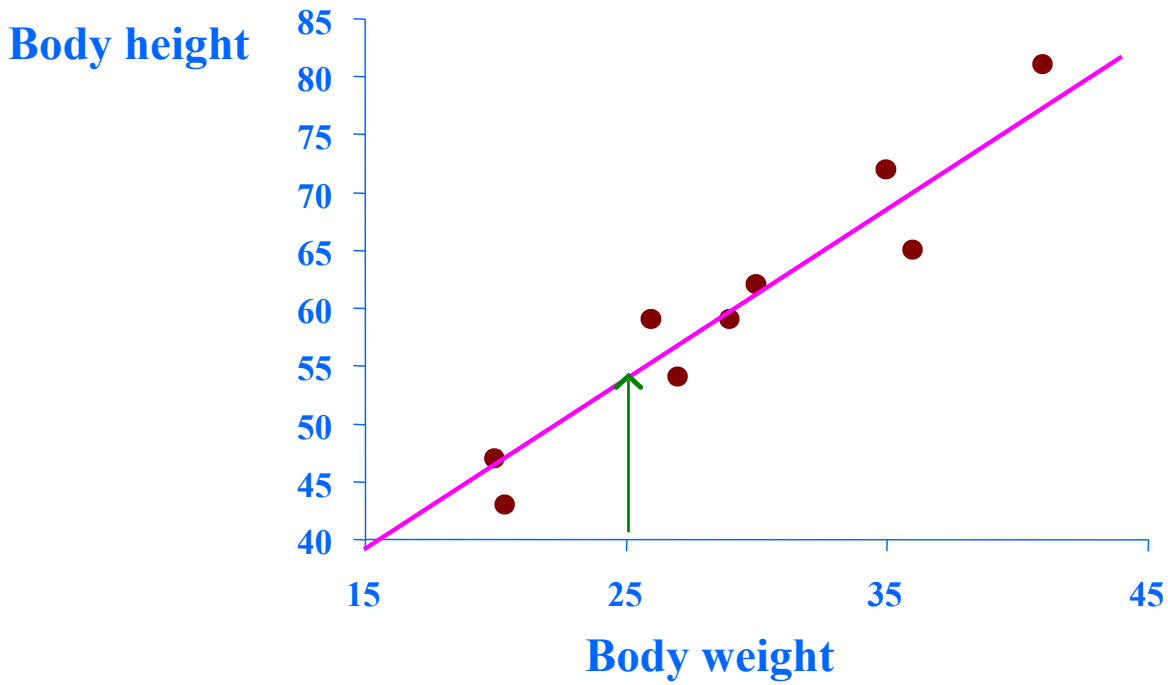


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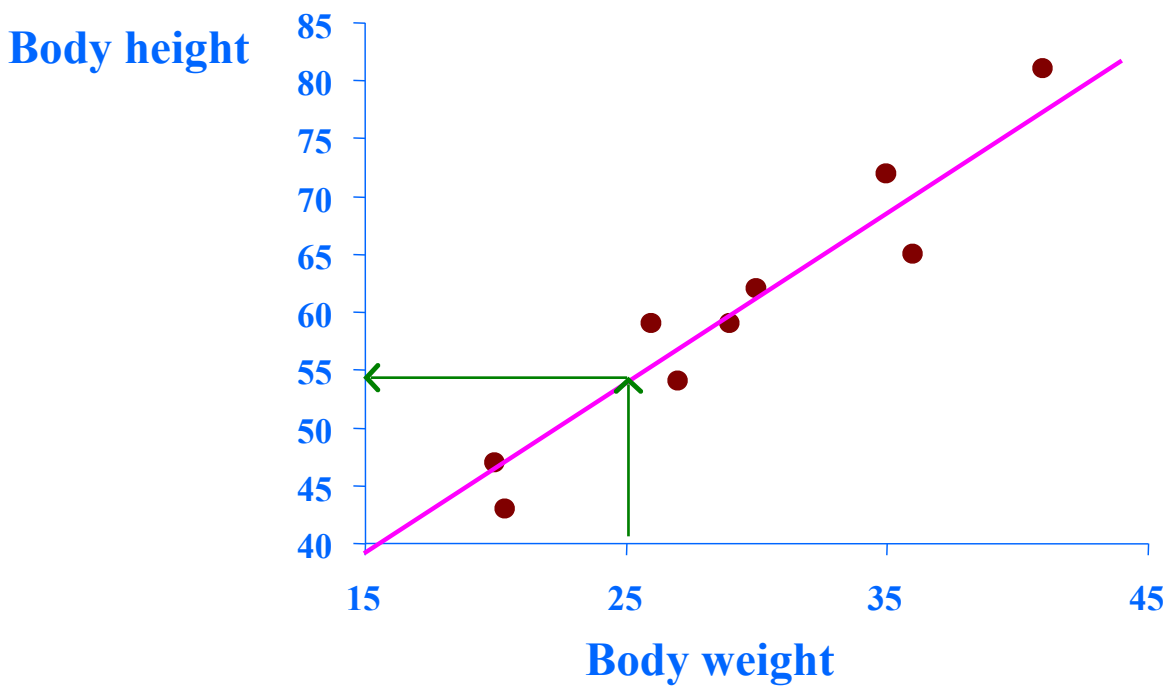


- express what we **don't know** (= "dependent variable")
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# Linear <u>Auto</u> Regression:

| *Time* | *Packets Sent(t)* |
|---|---|
| 1 | 43 |
| 2 | 54 |
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| … | … |
| N | **??** |

# Linear <u>Auto</u> Regression:

| *Time* | *Packets Sent (t-1)* | *Packets Sent(t)* |
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| 1 | - | 43 |
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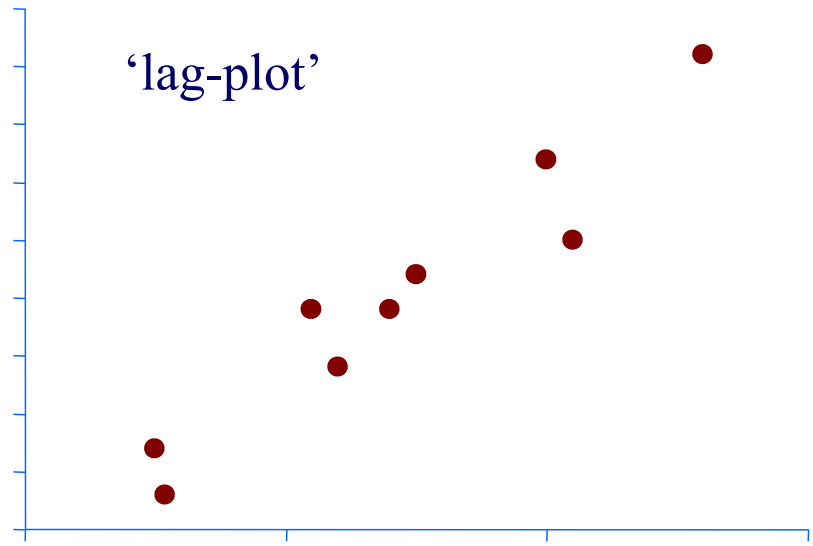
#packets sent at time **t**




#packets sent at time **t-1**

- lag $w = 1$
- <u>Dependent</u> variable = # of packets sent (S [t])
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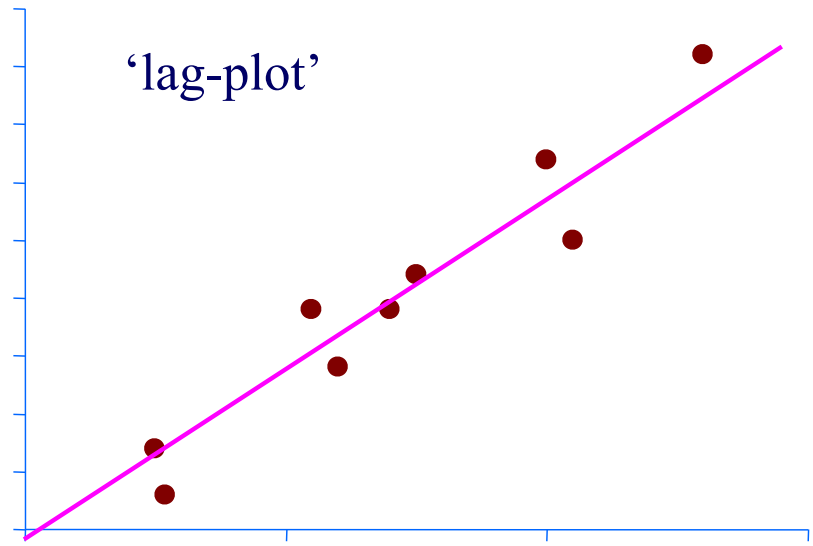


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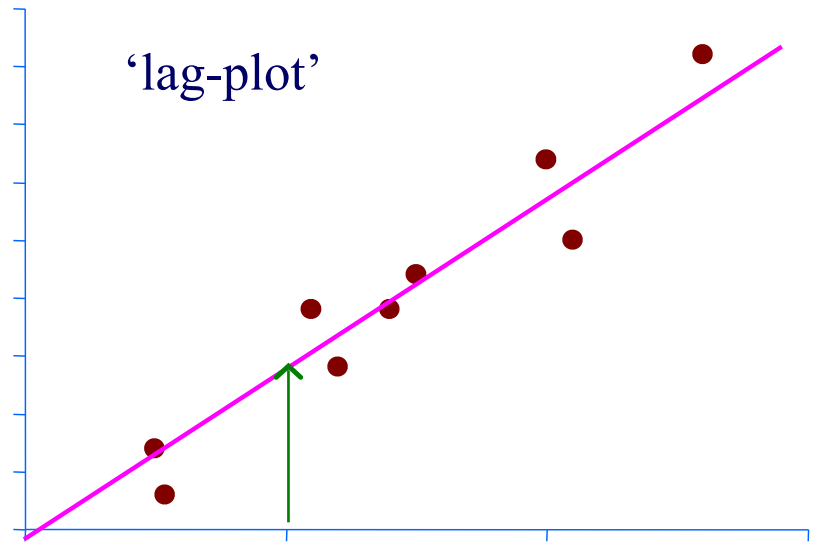


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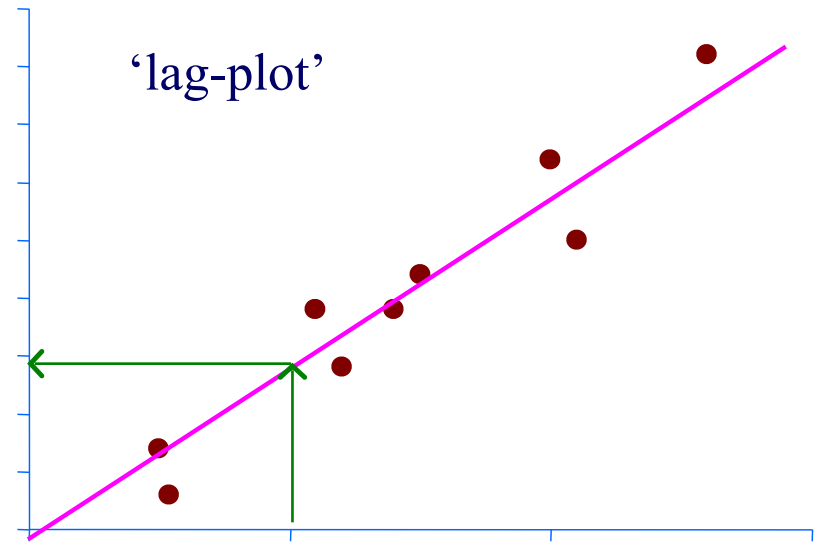


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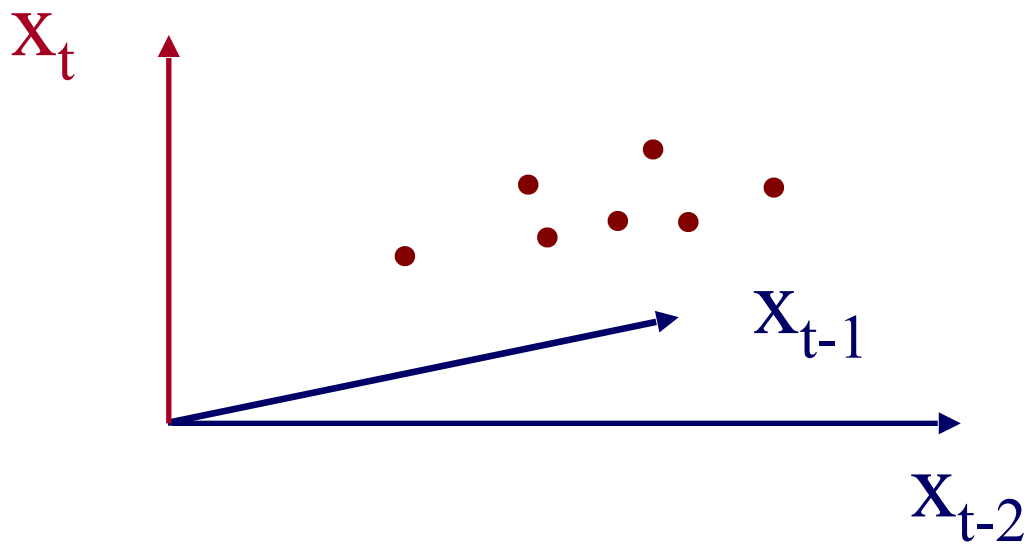
- Motivation
- ...
- Linear Forecasting
  - Auto-regression: **Least Squares; RLS**
  - Co-evolving time sequences
  - Examples
  - Conclusions

# More details:

- Q1: Can it work with window $w > 1$?
- A1: YES!



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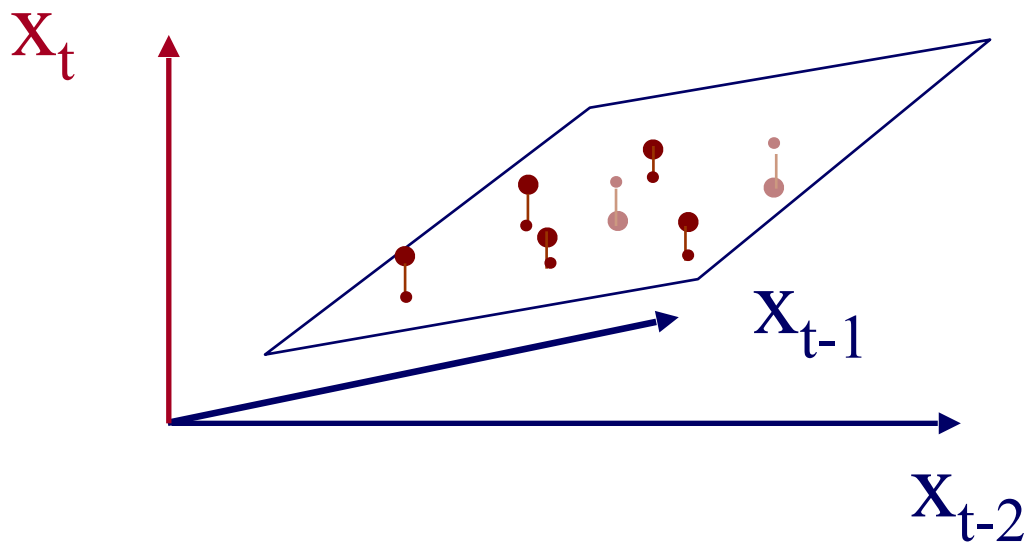

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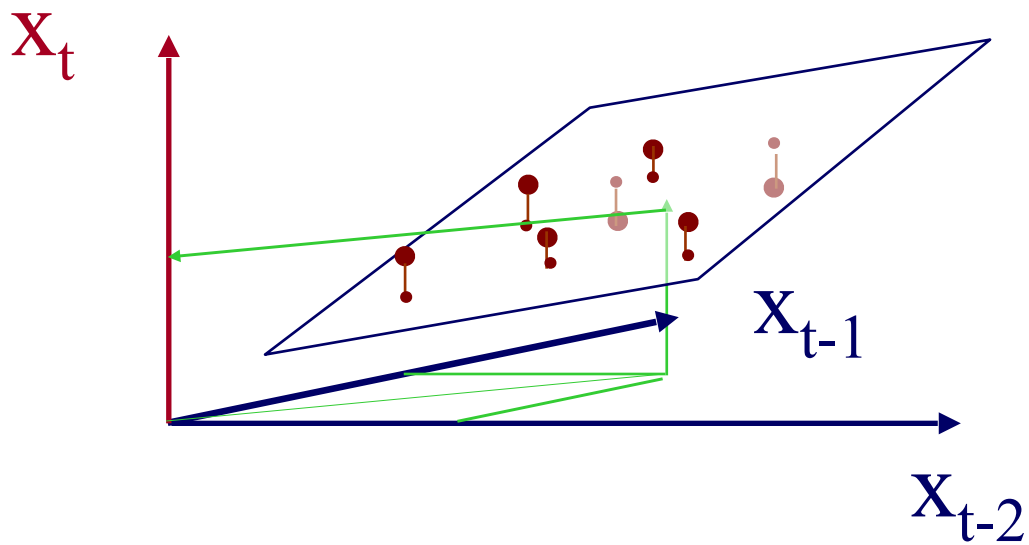


# More details:

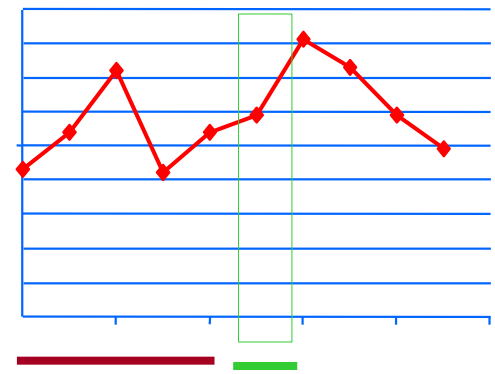
- Q1: Can it work with window $w > 1$?
- A1: YES! The problem becomes:

$$\mathbf{X}_{[\mathbf{N} \times \mathrm{w}]} \times \mathbf{a}_{[\mathbf{w} \times 1]} = \mathbf{y}_{[\mathbf{N} \times 1]}$$

- OVER-CONSTRAINED
  - **a** is the vector of the regression coefficients
  - **X** has the $N$ values of the $w$ indep. variables
  - **y** has the N values of the dependent variable

# More details:

- $\mathbf{X}_{[N \times w]} \times \mathbf{a}_{[w \times 1]} = \mathbf{y}_{[N \times 1]}$



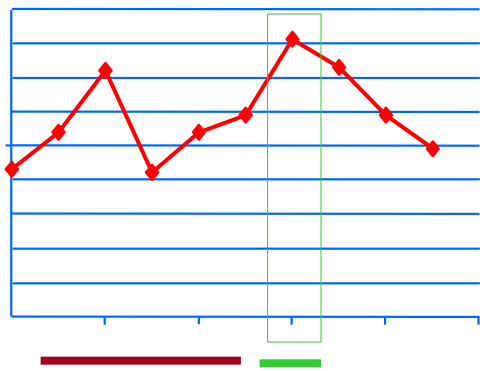
Ind-var1 Ind-var-w

time

$$\begin{bmatrix} X_{11}, X_{12}, \cdots, X_{1w} \\ X_{21}, X_{22}, \ldots, X_{2w} \\ \vdots \\ \vdots \\ \vdots \\ X_{N1}, X_{N2}, \ldots, X_{Nw} \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_w \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ y_N \end{bmatrix}$$

# More details:

- $\mathbf{X}_{[N \times w]} \times \mathbf{a}_{[w \times 1]} = \mathbf{y}_{[N \times 1]}$



Ind-var1 Ind-var-w

time

$$\begin{bmatrix} X_{11}, X_{12}, \cdots, X_{1w} \\ X_{21}, X_{22}, \ldots, X_{2w} \\ \vdots \\ \vdots \\ \vdots \\ X_{N1}, X_{N2}, \ldots, X_{Nw} \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_w \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ y_N \end{bmatrix}$$

# More details

- Q2: How to estimate $a_1, a_2, \ldots a_w = \mathbf{a}$?
- A2: with Least Squares fit

$$\boxed{\mathbf{a} = (\mathbf{X}^T \times \mathbf{X})^{-1} \times (\mathbf{X}^T \times \mathbf{y})}$$

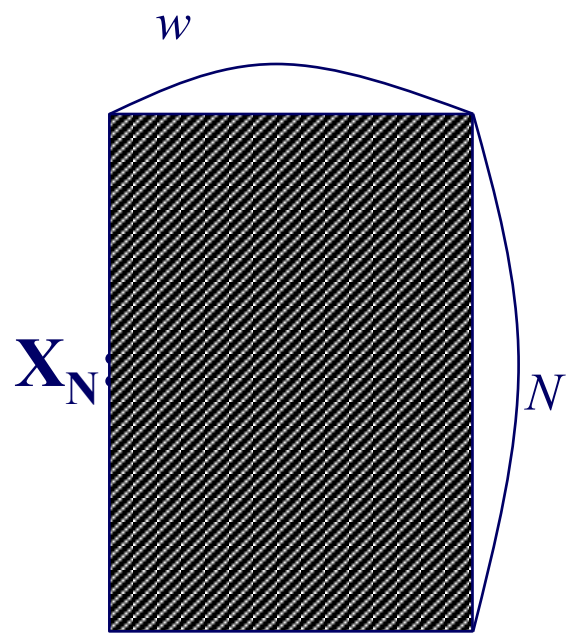
- (Moore-Penrose pseudo-inverse)
- **a** is the vector that minimizes the RMSE from **y**

# More details

- Straightforward solution:

$$\mathbf{a} = (\mathbf{X}^T \times \mathbf{X})^{-1} \times (\mathbf{X}^T \times \mathbf{y})$$

**a** : Regression Coeff. Vector
**X** : Sample Matrix




- Observations:
  - Sample matrix X grows over time
  - needs matrix inversion
  - $\mathbf{O}(N \times w^2)$ computation
  - $\mathbf{O}(N \times w)$ storage

# Even more details

- Q3: Can we estimate **a** incrementally?
- A3: Yes, with the brilliant, classic method of "Recursive Least Squares" (RLS) (see, e.g., [Yi+00], for details).
- We can do the matrix inversion, WITHOUT inversion! (How is that possible?!)

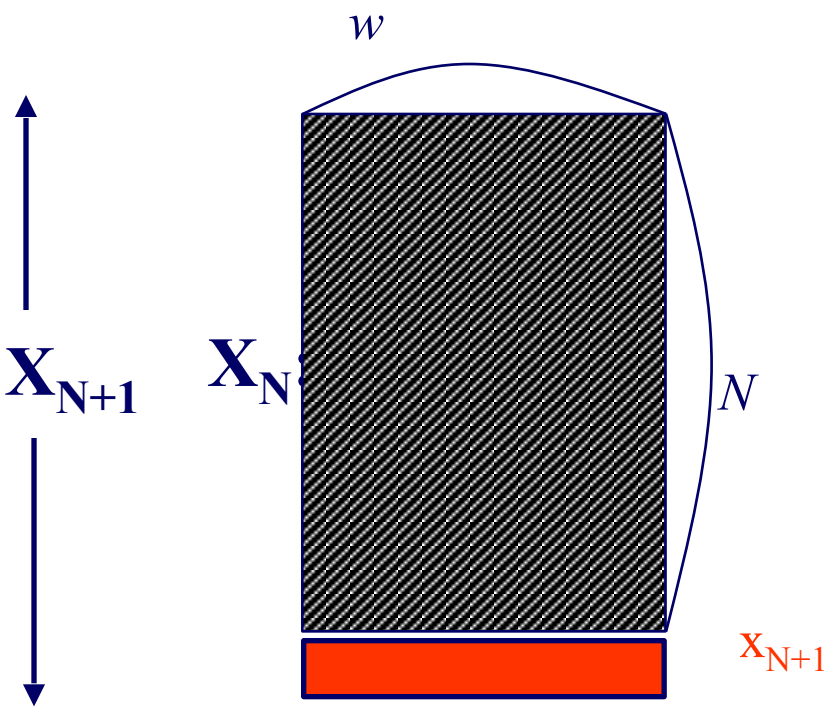
# Even more details

- Q3: Can we estimate **a** incrementally?
- A3: Yes, with the brilliant, classic method of **"Recursive Least Squares" (RLS)** (see, e.g., [Yi+00], for details).
- We can do the matrix inversion, WITHOUT inversion! (How is that possible?!)
- A: our matrix has special form: $(X^T X)$

SKIP

# More details

At the *N+1* time tick:



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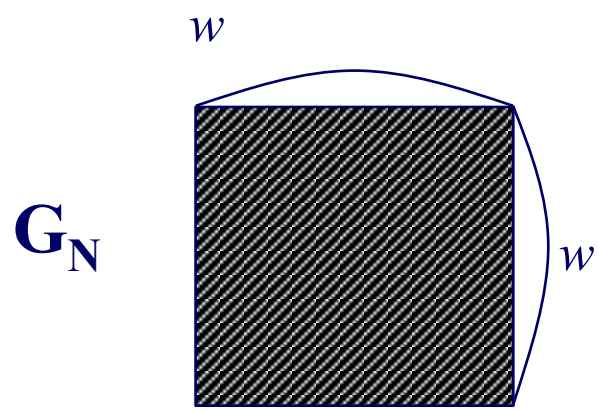
# More details

- Let $\mathbf{G}_N = ( \mathbf{X}_N^T \times \mathbf{X}_N )^{-1}$ ("gain matrix")
- $\mathbf{G}_{N+1}$ can be computed recursively from $\mathbf{G}_N$



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# EVEN more details:

$$G_{N+1} = G_N - [c]^{-1} \times [G_N \times x_{N+1}{}^T] \times x_{N+1} \times G_N$$

*1* x *w* row vector

$$c = [1 + x_{N+1} \times G_N \times x_{N+1}{}^T]$$

Let's elaborate
(VERY IMPORTANT, VERY VALUABLE!)

SKIP

# EVEN more details:

$$a = [X_{N+1}{}^{T} \times X_{N+1}]^{-1} \times [X_{N+1}{}^{T} \times y_{N+1}]$$

SKIP

# EVEN more details:

$$a = [X_{N+1}{}^{T} \times X_{N+1}]^{-1} \times [X_{N+1}{}^{T} \times y_{N+1}]$$

[w x 1]

[(N+1) x w]

[(N+1) x 1]

[w x (N+1)]

[w x (N+1)]

SKIP

# EVEN more details:

$$a = \boxed{[X_{N+1}{}^{T} \times X_{N+1}]^{-1}} \times [X_{N+1}{}^{T} \times y_{N+1}]$$

[(N+1) x w]

[w x (N+1)]

SKIP

# EVEN more details:

$$a = [X_{N+1}{}^{T} \times X_{N+1}]^{-1} \times [X_{N+1}{}^{T} \times y_{N+1}]$$

'gain matrix'

1 x w row vector

$$G_{N+1} \equiv [X_{N+1}{}^{T} \times X_{N+1}]^{-1}$$

$$G_{N+1} = G_N - [c]^{-1} \times [G_N \times x_{N+1}{}^{T}] \times x_{N+1} \times G_N$$

$$c = [1 + x_{N+1} \times G_N \times x_{N+1}{}^{T}]$$

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# EVEN more details:

$$G_{N+1} = G_N - [c]^{-1} \times [G_N \times x_{N+1}{}^T] \times x_{N+1} \times G_N$$

$$c = [1 + x_{N+1} \times G_N \times x_{N+1}{}^T]$$

SKIP

# EVEN more details:

1x1

1xw

wxw wxw wxw wx1 wxw

$$G_{N+1} = G_N - [c]^{-1} \times [G_N \times x_{N+1}{}^T] \times x_{N+1} \times G_N$$

**SCALAR!**

$$c = [1 + x_{N+1} \times G_N \times x_{N+1}{}^T]$$

SKIP

# Altogether:

$$a = [X_{N+1}{}^{T} \times X_{N+1}]^{-1} \times [X_{N+1}{}^{T} \times y_{N+1}]$$

$$G_{N+1} \equiv [X_{N+1}{}^{T} \times X_{N+1}]^{-1}$$

$$G_{N+1} = G_N - [c]^{-1} \times [G_N \times x_{N+1}{}^{T}] \times x_{N+1} \times G_N$$

$$c = [1 + x_{N+1} \times G_N \times x_{N+1}{}^{T}]$$

SKIP

# Altogether:

$$G_0 \equiv \delta \; I$$

where
I: w x w identity matrix
δ: a large positive number

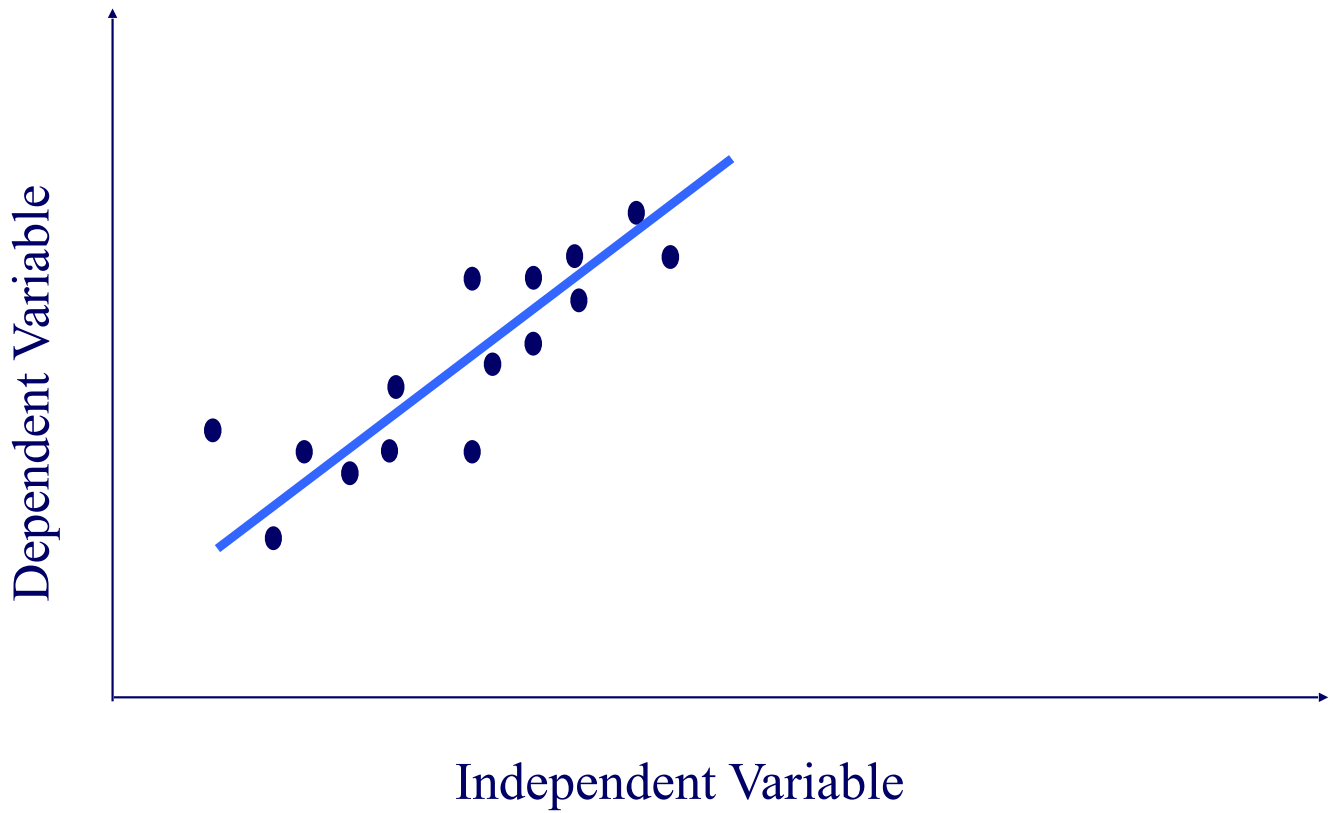
# Comparison:

- Straightforward Least Squares
  - Needs huge matrix (**growing** in size) $O(N \times w)$
  - Costly matrix operation $O(N \times w^2)$
- Recursive LS
  - Need much smaller, fixed size matrix $O(w \times w)$
  - Fast, incremental computation $O(1 \times w^2)$
  - **no matrix inversion**

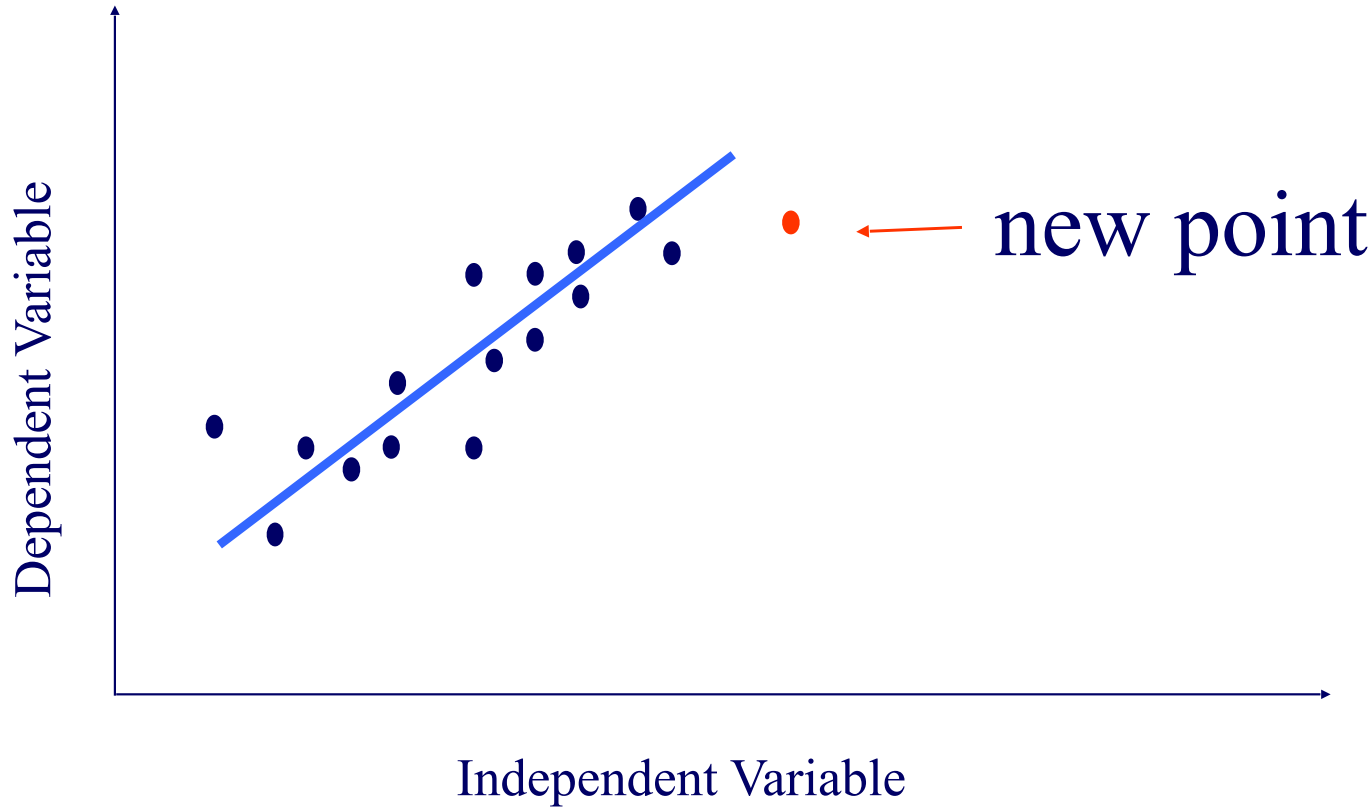
$N = 10^6, \quad w = 1\text{-}100$

# Pictorially:

- Given:

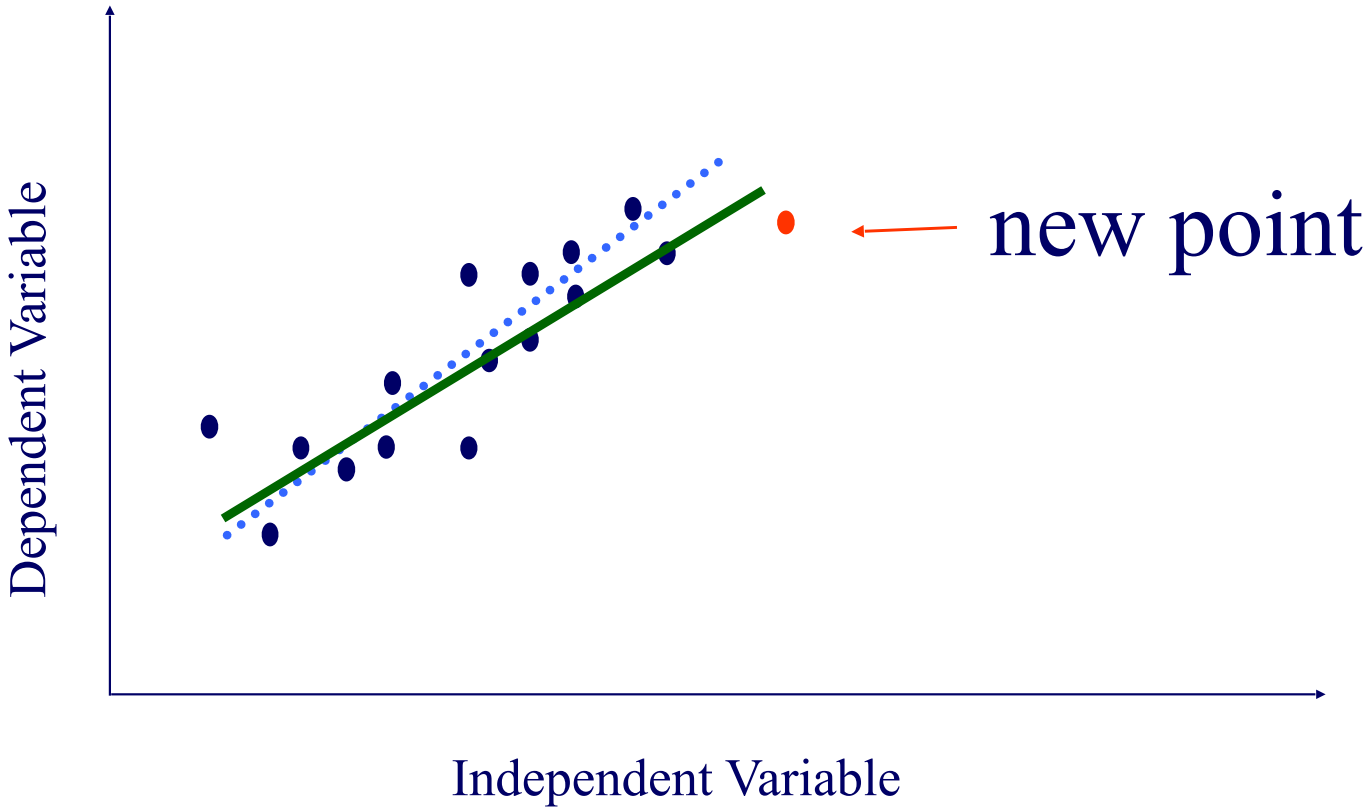


# Pictorially:



# Pictorially:

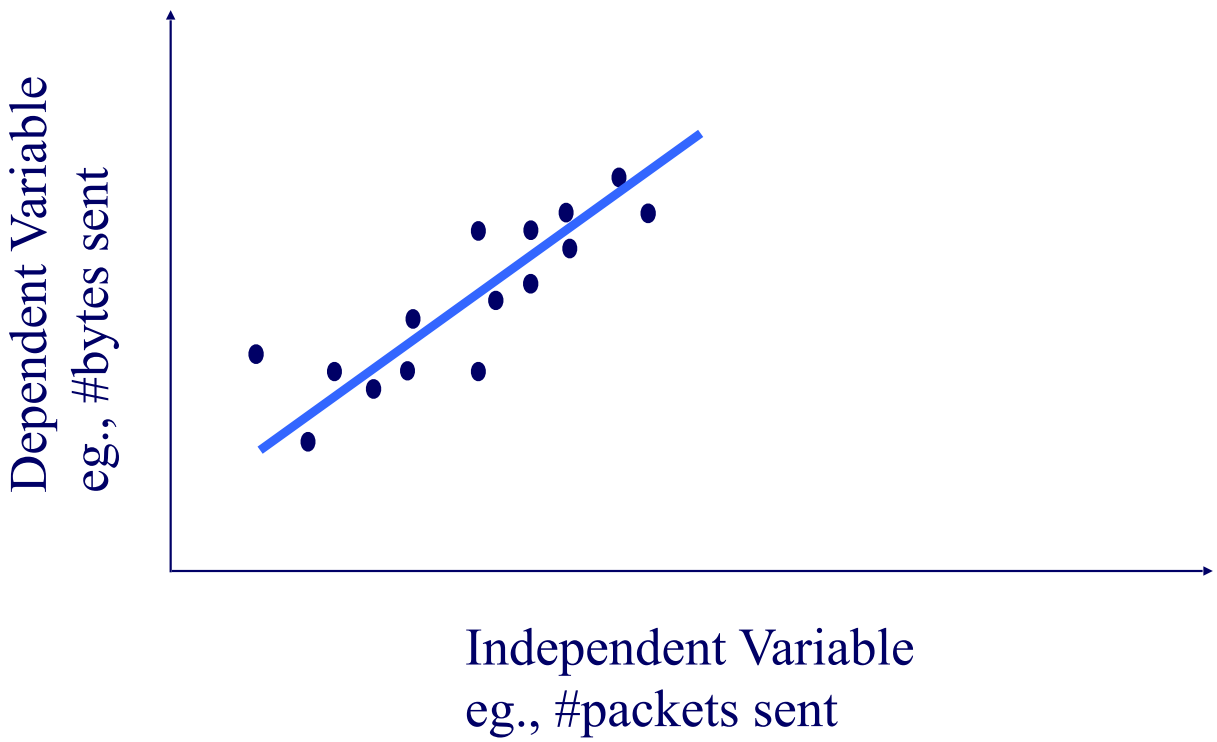
RLS: quickly compute new best fit



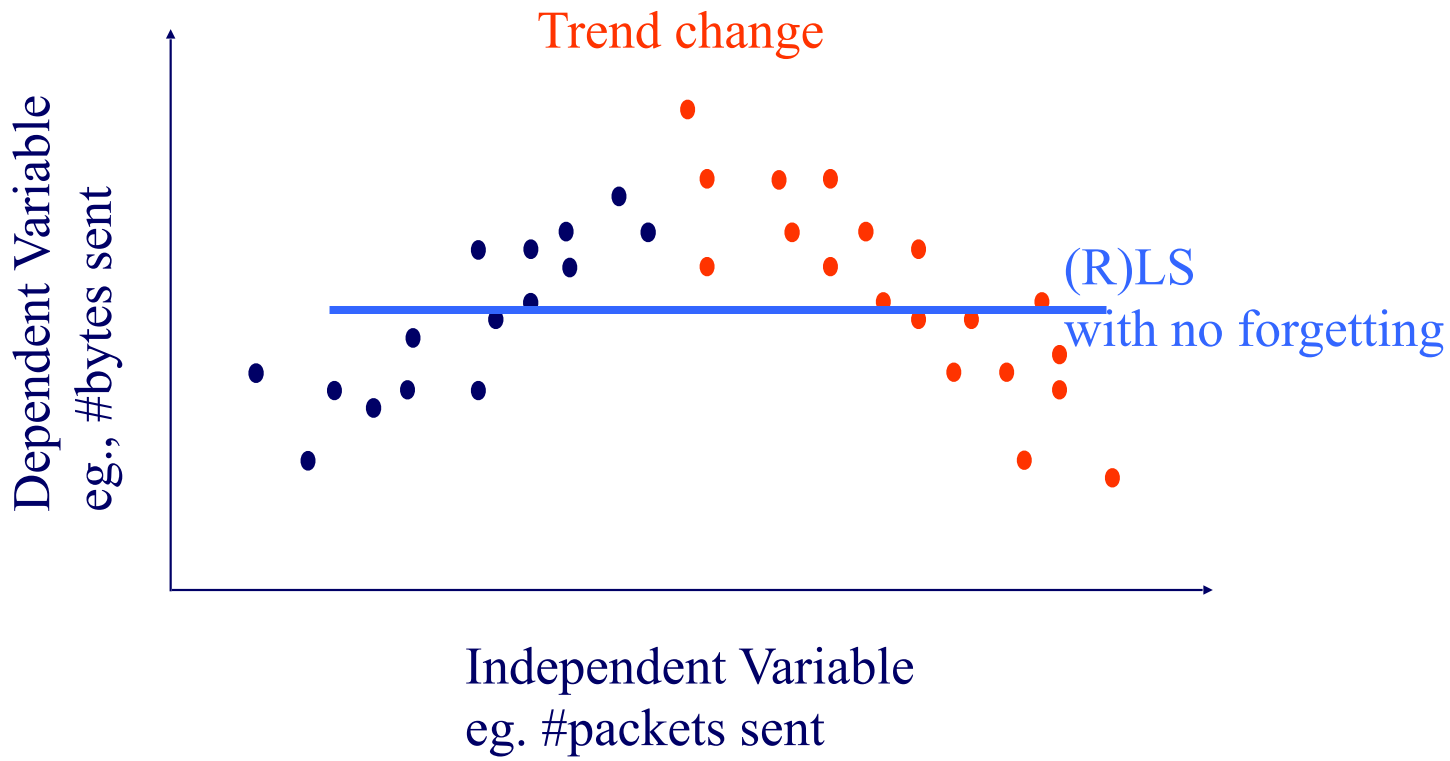
# Even more details

- Q4: can we ‘forget’ the older samples?
- A4: Yes - RLS can easily handle that [Yi+00]:

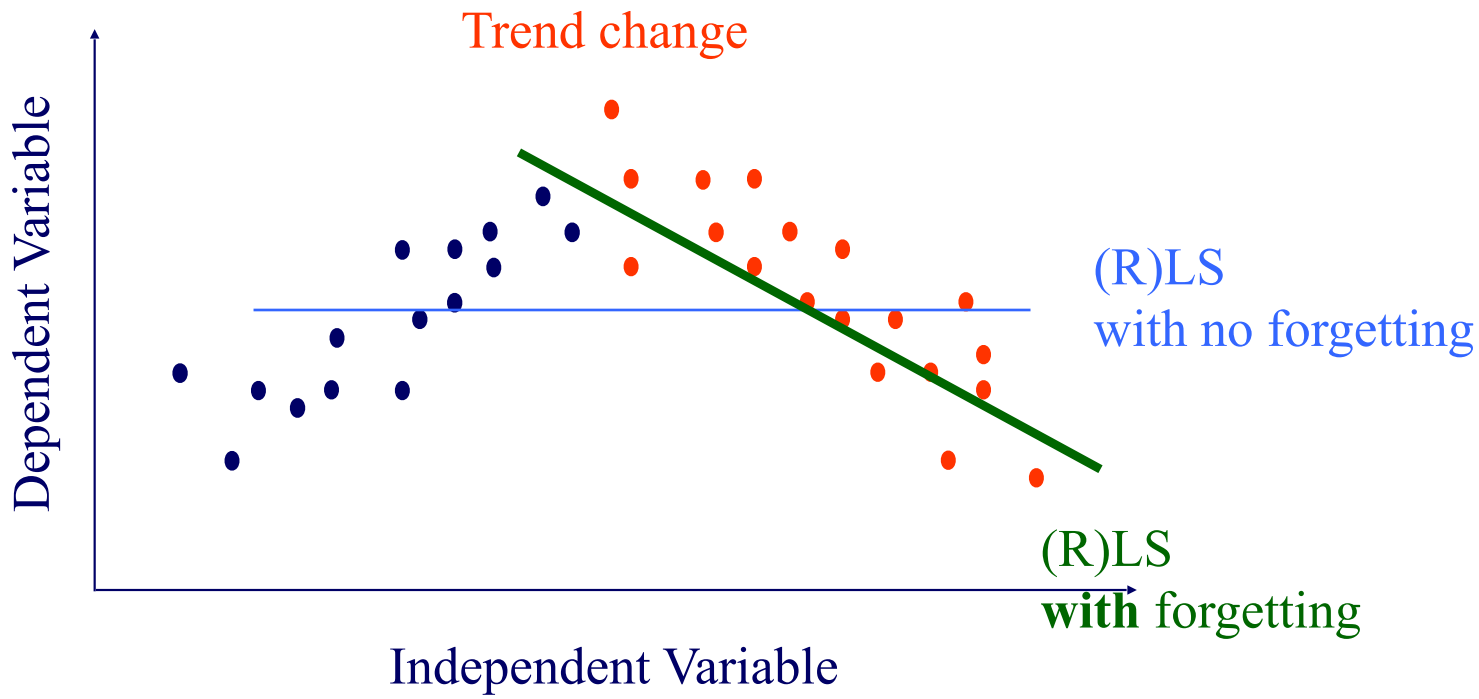
# Adaptability - ‘forgetting’



# Adaptability - ‘forgetting’



# Adaptability - ‘forgetting’




- RLS: can *trivially* handle ‘forgetting’