Graphs II
Centrality, and algorithms you should know

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Partly based on materials by Professors Guy Lebanon, Jeffrey Heer, John Stasko, Christos Faloutsos, Le Song
Centrality
=
“Importance”
Why Node Centrality?

What can we do if we can rank all the nodes in a graph (e.g., Facebook, LinkedIn, Twitter)?

• Find **celebrities** or influential people in a social network (Twitter)

• Find “**gatekeepers**” who connect communities (headhunters love to find them on LinkedIn)

• What else?
More generally

Helps graph analysis, visualization, understanding, e.g.,

- let us rank nodes, group or study them by centrality
- only show subgraph formed by the top 100 nodes, out of the millions in the full graph
- similar to google search results (ranked, and they only show you 10 per page)

Can also compute edge centrality. Here we focus on node centrality.
Degree Centrality (easiest)

Degree = number of neighbors

For directed graphs

• in degree = # incoming edges
• out degree = # outgoing edges

Algorithms?

• Sequential scan through edge list
• What about for a graph stored in SQLite?
Computing degrees using SQL

Recall simplest way to store a graph in SQLite:

```
edges(source_id, target_id)
```

1. Create index for each column
2. Use **group by** statement to find node degrees

```
select count(*) from edges group by source_id;
```
Betweenness Centrality

High betweenness

• = important “gatekeeper” or liaison

A node’s betweenness

• = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}

Needs to compute all-pairs shortest path (O(N^3)). Slow.
Clustering Coefficient

Technically not a centrality measure, but useful

A node’s clustering coefficient is a measure of how close the node’s neighbors are from forming a clique.

- Value of 1 = neighbors form a clique
- Value of 0 = No edges among neighbors

(Assuming undirected graph)
Computing Clustering Coefficient...

Requires *triangle counting*

Real social networks have a lot of triangles

- Friends of friends are friends

But: triangles are *expensive* to compute

(3-way join; several approx. algos)

Can we do that quickly?
Super Fast Triangle Counting
[Tsourakakis ICDM 2008]

But: triangles are expensive to compute
(3-way join; several approx. algos)
Q: Can we do that quickly?
A: Yes!

#triangles = \( \frac{1}{6} \sum (\lambda_i^3) \)

(and, because of skewness, we only need the top few eigenvalues!)
Wikipedia graph 2006-Nov-04
≈ 3.1M nodes ≈ 37M edges

1000x+ speed-up, >90% accuracy
PageRank (google)

Problem: PageRank

Given a directed graph, find its most interesting/central node

A node is important, if it is connected with important nodes (recursive, but OK!)
Problem: PageRank - solution

Given a directed graph, find its most interesting/central node

Proposed solution: Random walk; spot most ‘popular’ node (-> steady state prob. (ssp))

“state” = webpage

A node has high ssp, if it is connected with high ssp nodes (recursive, but OK!)
(Simplified) PageRank

• Let $A$ be the adjacency matrix;

• let $B$ be the transition matrix: transpose, column-normalized - then

$$
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array}
\begin{array}{c}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5
\end{array}
\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1/2 & 1/2 \\
0 & 1/2 & 1/2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
= 
\begin{array}{c}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5
\end{array}
$$
(Simplified) PageRank

- $B \mathbf{p} = \mathbf{p}$
(Simplified) PageRank

- $B \ p = 1 \ * \ p$
- thus, $p$ is the eigenvector that corresponds to the highest eigenvalue ($=1$, since the matrix is column-normalized)
- Why does such a $p$ exist?
  - $p$ exists if $B$ is nxn, nonnegative, irreducible ([Perron–Frobenius theorem])
(Simplified) PageRank

- B p = 1 * p
- thus, p is the eigenvector that corresponds to the highest eigenvalue (=1, since the matrix is column-normalized)
- Why does such a p exist?
  - p exists if B is nxn, nonnegative, irreducible
    [Perron–Frobenius theorem]
(Simplified) PageRank

- In short: imagine a particle randomly moving along the edges
- compute its steady-state probabilities (ssp)

Full version of algo:
with occasional random jumps
Why? To make the matrix irreducible
Full Algorithm

• With probability $1-c$, fly-out to a random node
• Then, we have

\[ p = c \mathbf{B} p + \frac{1-c}{n} \mathbf{1} \Rightarrow \]
\[ p = \frac{1-c}{n} [\mathbf{I} - c \mathbf{B}]^{-1} \mathbf{1} \]
Full Algorithm

- With probability $1 - c$, fly-out to a random node
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  \[ p = c \mathbf{B} p + \frac{1-c}{n} \mathbf{1} \Rightarrow \]
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Alternative notation – eigenvector viewpoint

\[ M \quad \text{Modified transition matrix} \]

\[ M = cB + \frac{(1-c)}{n} \begin{bmatrix} 1 & 1^T \end{bmatrix} \]

Then

\[ p = Mp \]

That is: the steady state probabilities = PageRank scores form the first eigenvector of the ‘modified transition matrix’
PageRank for graphs (generally)

You can compute PageRank for any graphs

Should be in your algorithm “toolbox”

- Better than simple centrality measure (e.g., degree)
- Fast to compute for large graphs (O(E))

But can be “misled” (Google Bomb)

- How?
Personalized PageRank

Make one small variation of PageRank

• Intuition: not all pages are equal, some more relevant to a person’s specific needs

• How?
“Personalizing” PageRank

- With probability $1-c$, fly-out to a random node some preferred nodes
- Then, we have
  \[ p = c B p + \frac{1-c}{n} \mathbf{1} \Rightarrow \]
  \[ p = \frac{1-c}{n} \left[ I - c B \right]^{-1} \mathbf{1} \]
Why learn about Personalized PageRank?

Can be used for recommendation, e.g.,

• If I like this webpage, what would I also be interested?

• If I like this product, what other products I also like? (in a user-product bipartite graph)

Again, very flexible. Can be run on any graph

Will see some interactive tools in next lecture that uses Personalized PageRank